Off-the-Grid Compressive Imaging: Recovery of Piecewise Constant Images from Few Fourier Samples

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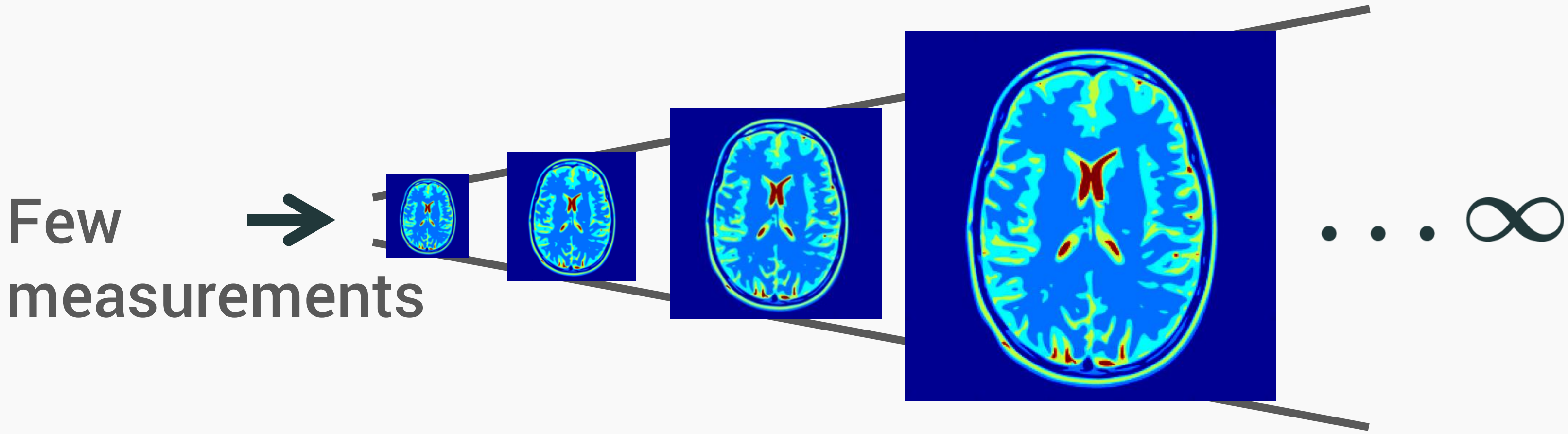
Our goal is to develop theory and algorithms for compressive off-the-grid imaging

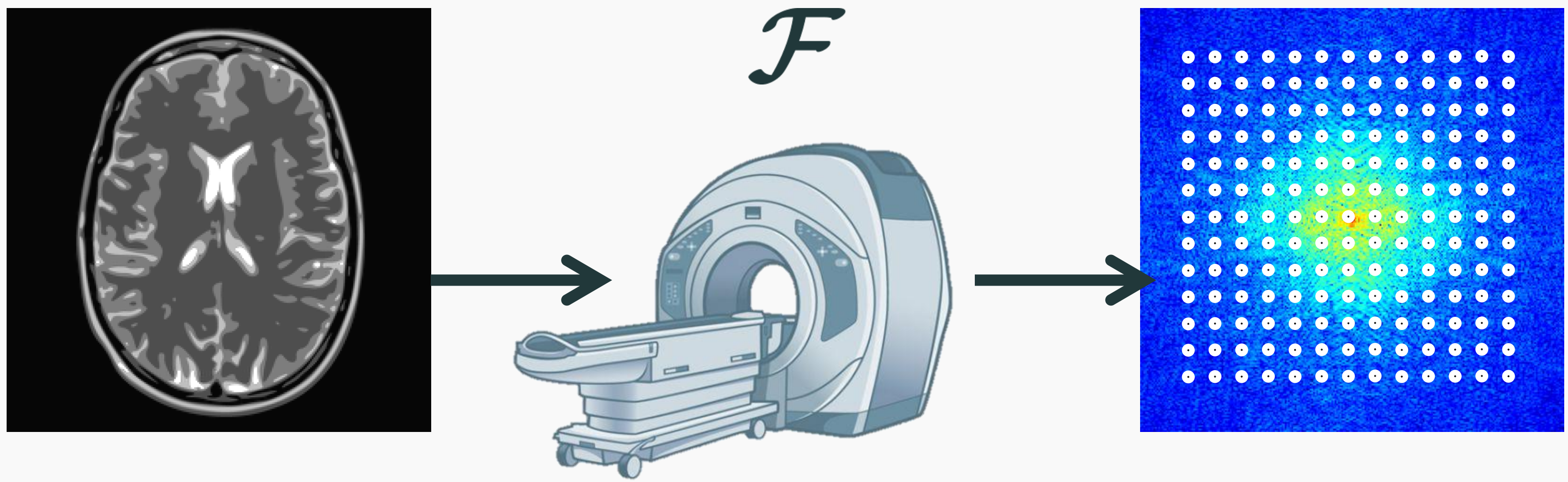


Off-the-grid = Continuous domain representation

Compressive off-the-grid *imaging*:

Exploit continuous domain modeling to improve image recovery from few measurements

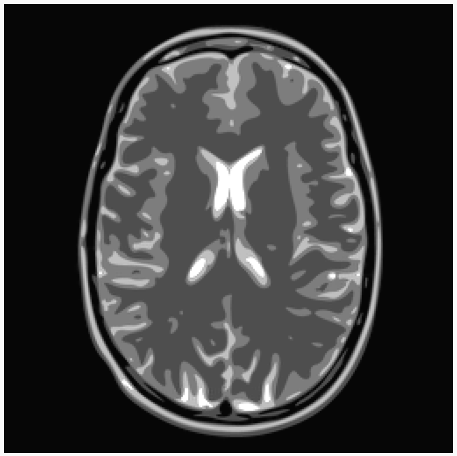
Motivation: MRI Reconstruction



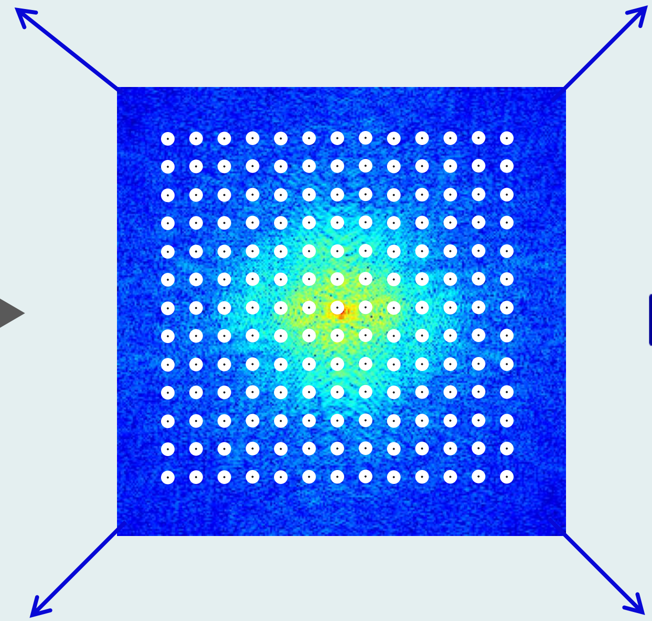
Main Problem:

Reconstruct image from Fourier domain samples

Related: Computed Tomography, Florescence Microscopy



$$\xrightarrow{\mathcal{F}}$$



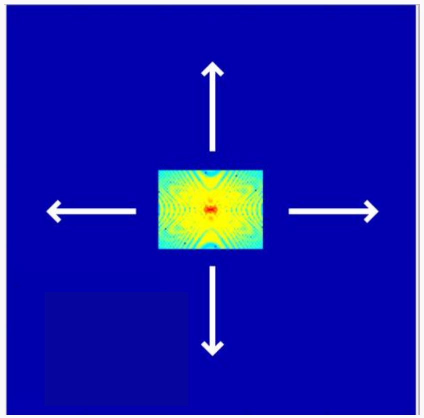
$$\mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in [0, 1]^{\mathbf{d}}$$

$$\widehat{\mathbf{f}}[\mathbf{k}] := \int_{[0,1]^{\mathbf{d}}} \mathbf{f}(\mathbf{x}) e^{-j2\pi \mathbf{k}\cdot\mathbf{x}} \mathbf{dx}$$

Uniform Fourier Samples =
Fourier Series Coefficients

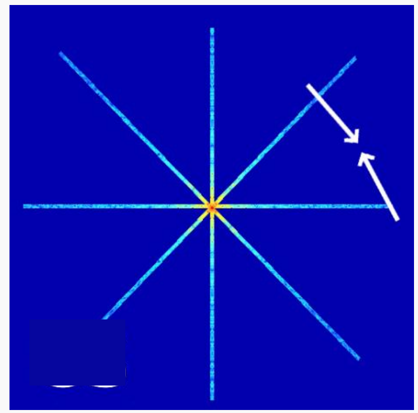
Types of “Compressive” Fourier Domain Sampling

low-pass

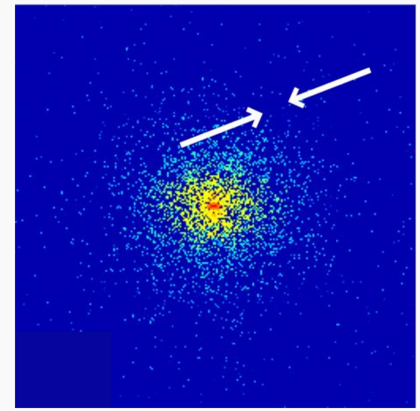


vs.

radial



random



Fourier Extrapolation

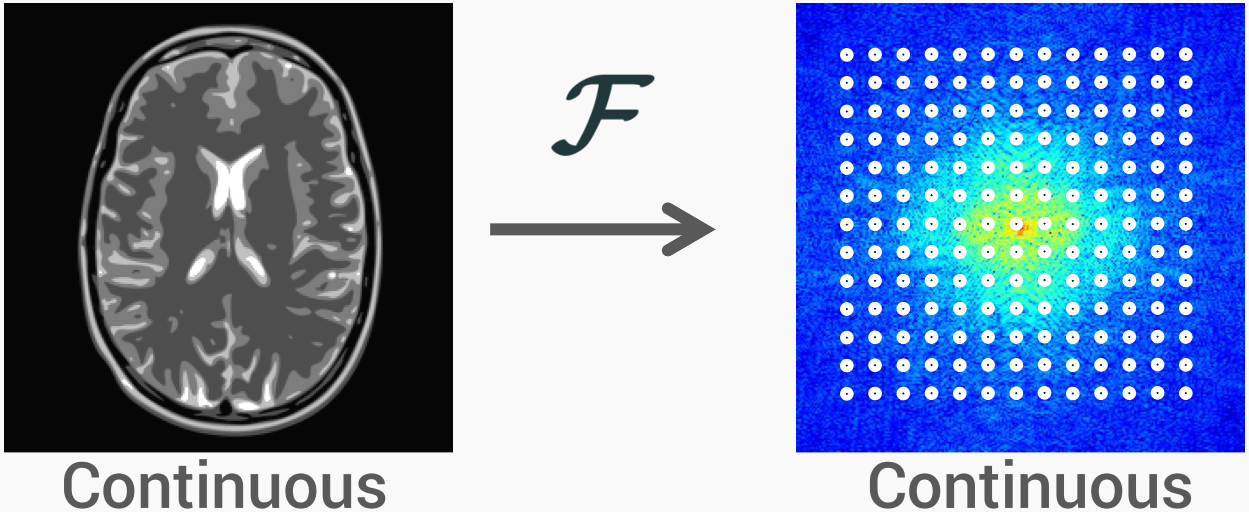
Fourier Interpolation

Super-resolution recovery

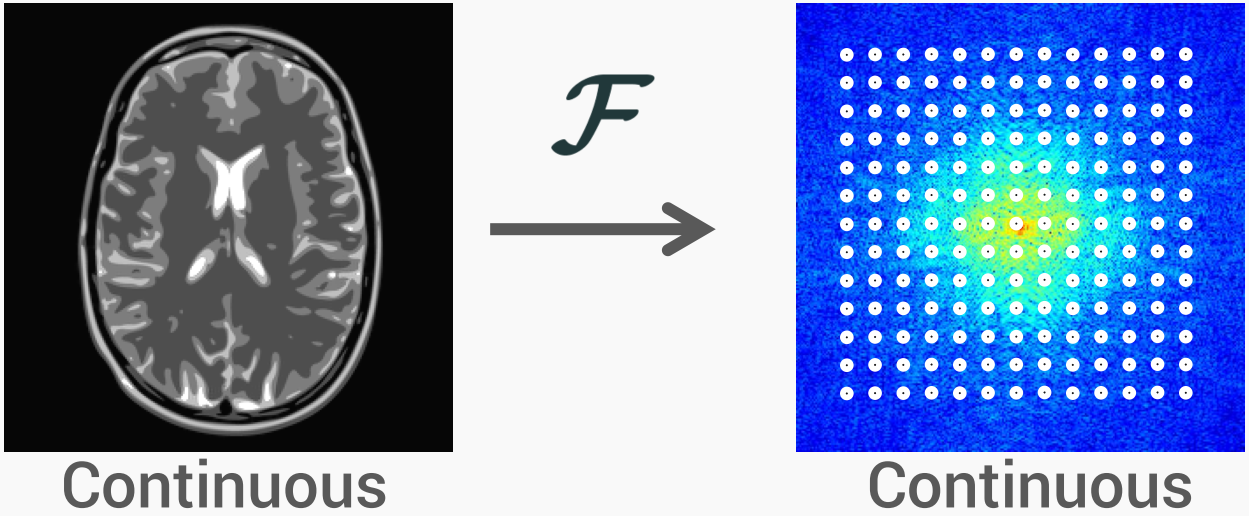
“Compressed Sensing” recovery

CURRENT DISCRETE PARADIGM

“True” measurement model:

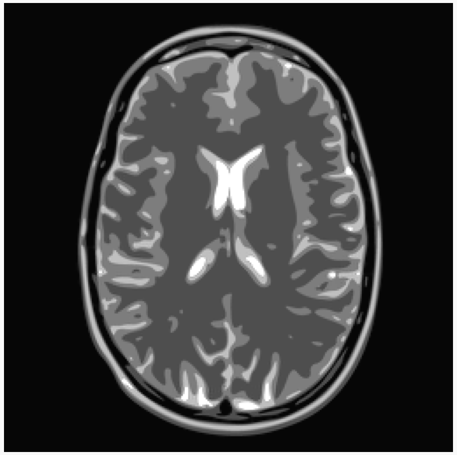


“True” measurement model:



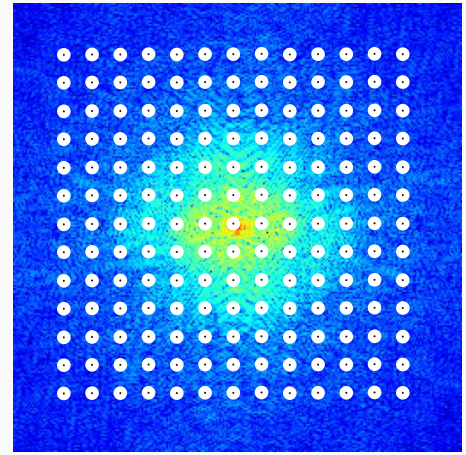
Approximated measurement model:

DFT Reconstruction



Continuous

$\xrightarrow{\mathcal{F}}$



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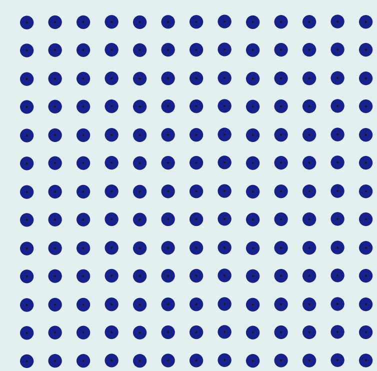
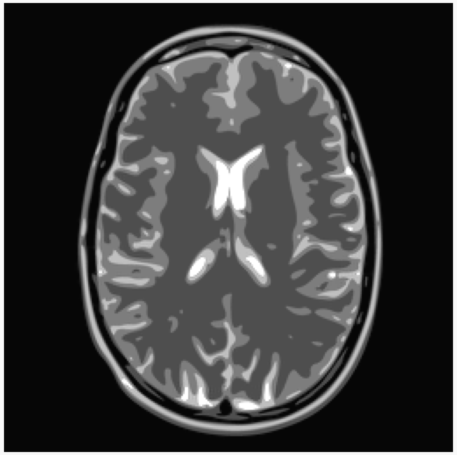
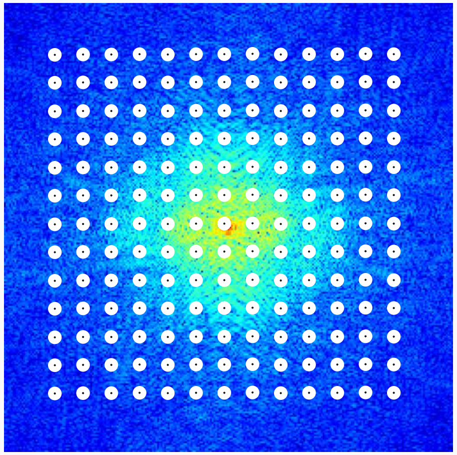


DFT Reconstruction

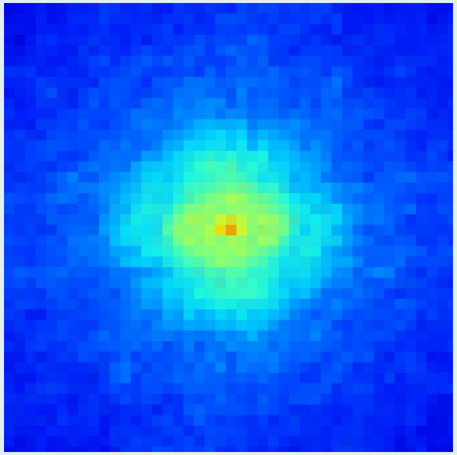


Continuous

$\xrightarrow{\mathcal{F}}$

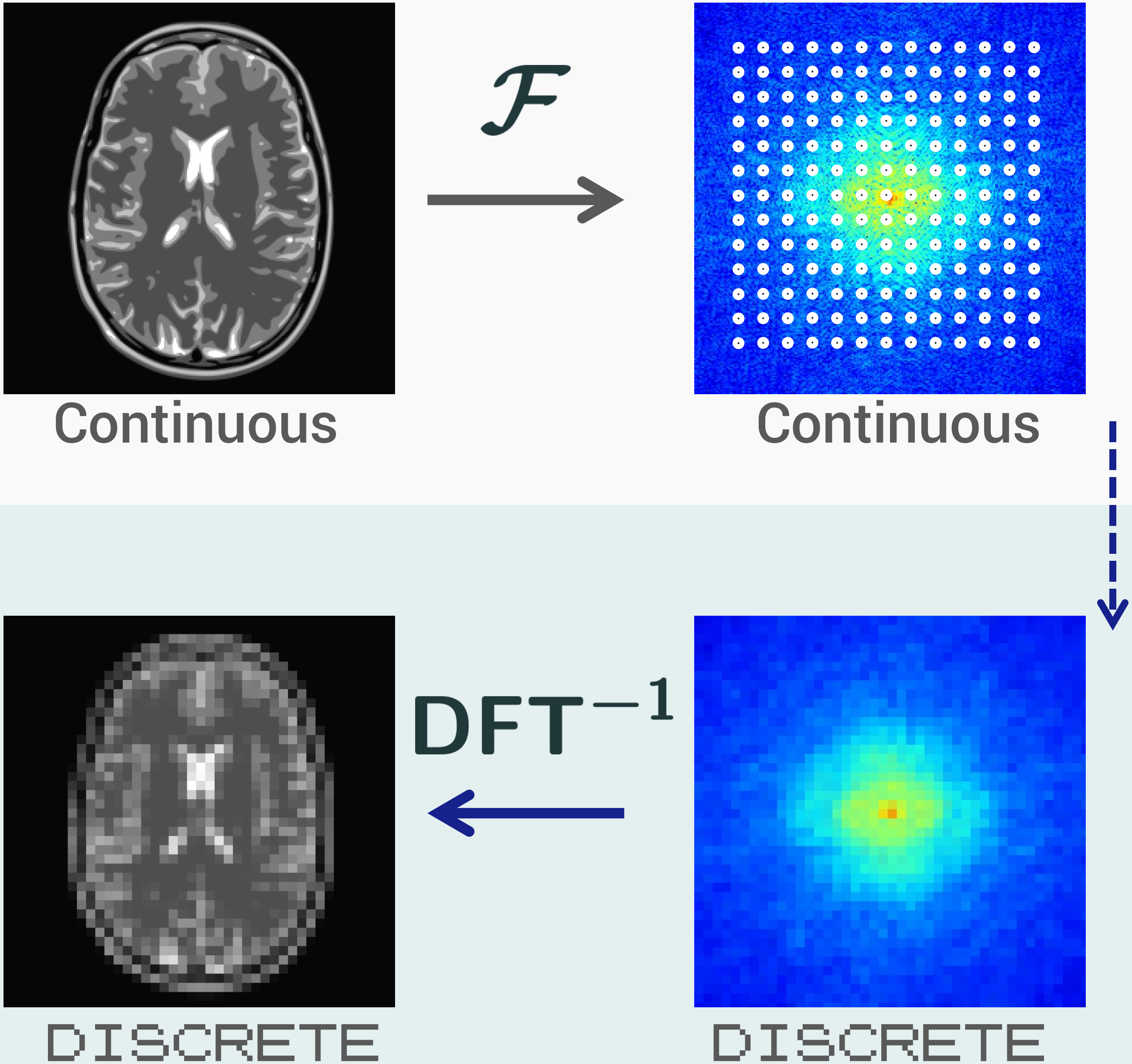


Continuous

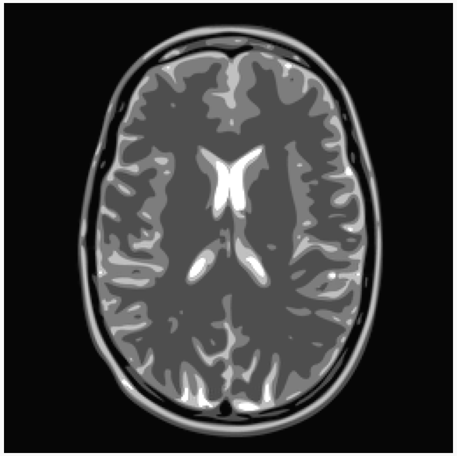


DISCRETE

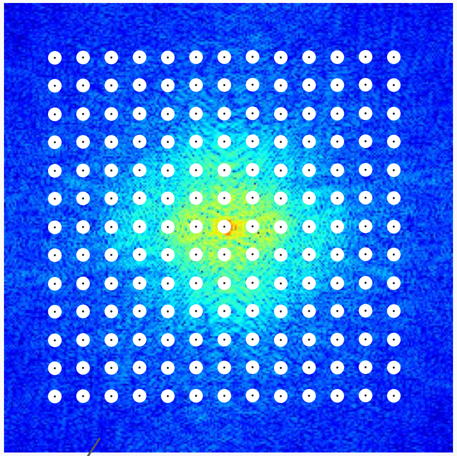
DFT Reconstruction



"Compressed Sensing" Recovery

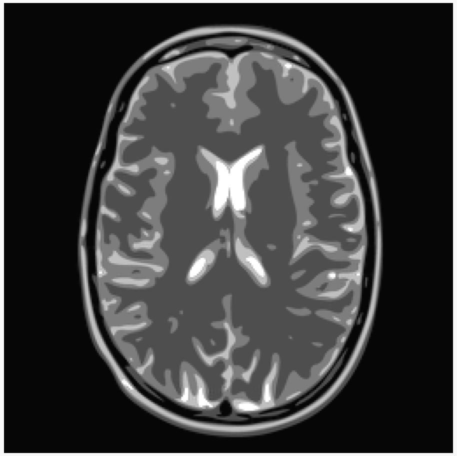


$\xrightarrow{\mathcal{F}}$



Full sampling is costly!
(or impossible—e.g. Dynamic MRI)

"Compressed Sensing" Recovery



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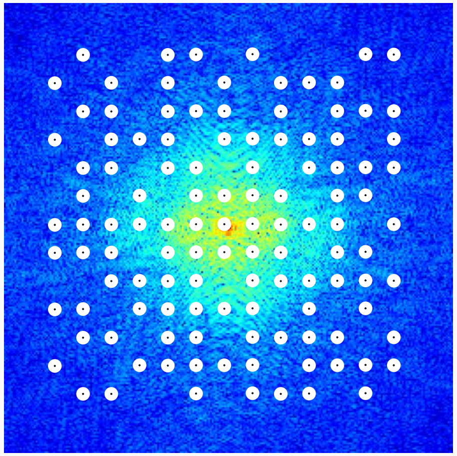
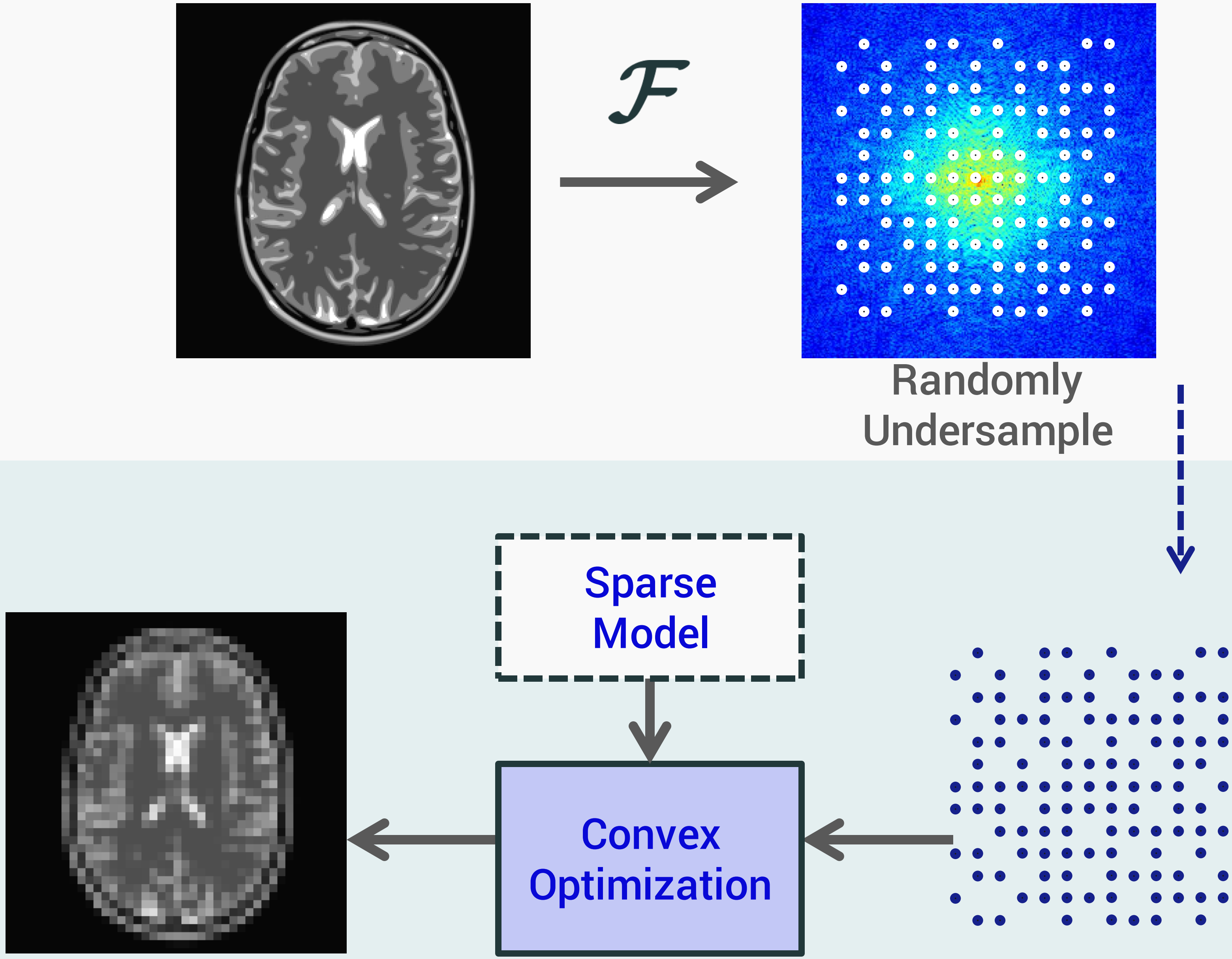


Randomly
Undersample

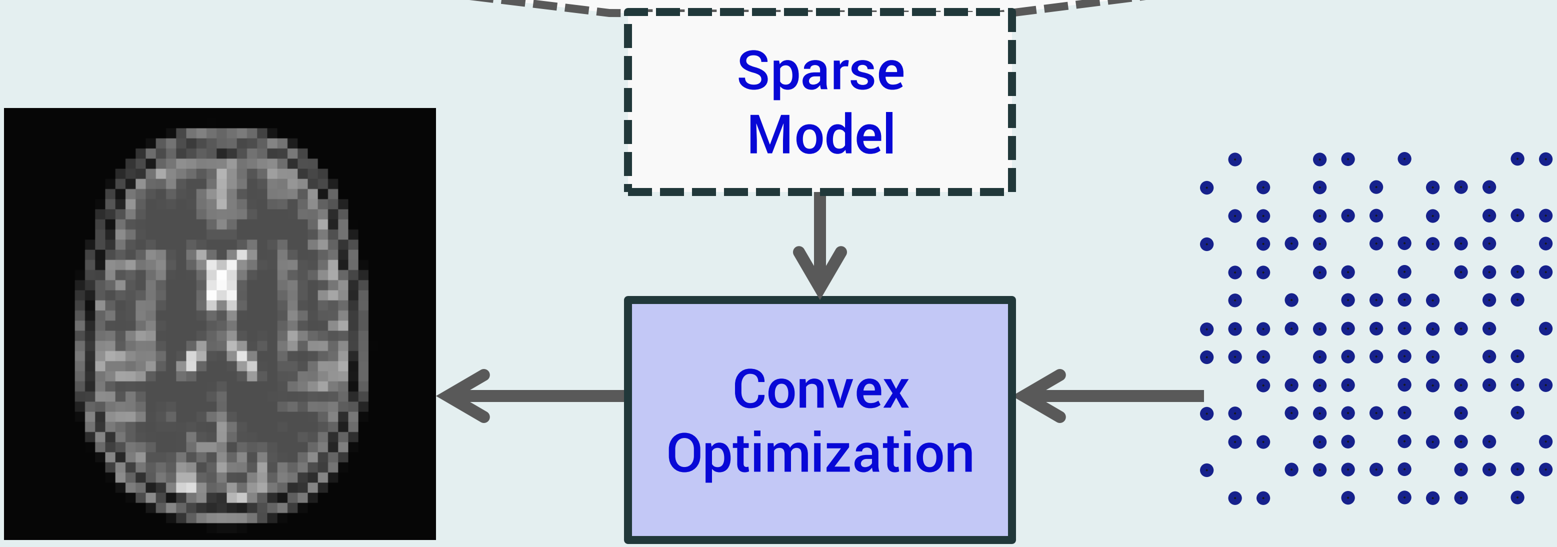
“Compressed Sensing” Recovery



Example:
Assume discrete gradient
of image is sparse

↕

Piecewise constant model



Recovery by Total Variation (TV) minimization

TV semi-norm: $\|\mathbf{g}\|_{\mathrm{TV}} = \sum_{\mathbf{i},\mathbf{j}} \sqrt{|\mathbf{g}_{\mathbf{i}+1,\mathbf{j}} - \mathbf{g}_{\mathbf{i},\mathbf{j}}|^2 + |\mathbf{g}_{\mathbf{i},\mathbf{j}+1} - \mathbf{g}_{\mathbf{i},\mathbf{j}}|^2}$

i.e., L1-norm of discrete gradient magnitude

$\sum_{\mathbf{i},\mathbf{j}}$ 

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$$\min_{\mathbf{g}\in\mathbb{C}^{N\times N}} \|\mathbf{g}\|_{TV} \quad \text{subject to} \quad \mathbf{F}_{\Omega}\mathbf{g} = \mathbf{F}_{\Omega}\mathbf{f} \qquad (\mathbf{TV}\text{-min})$$

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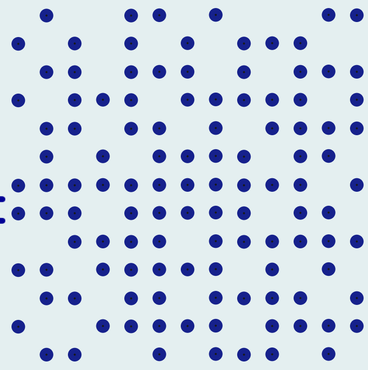
Restricted DFT

$\Omega =$ 

Sample locations

Recovery by Total Variation (TV) minimization

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Restricted DFT

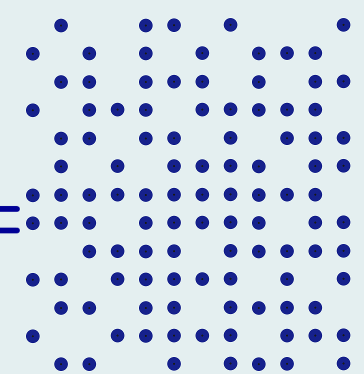
Convex optimization problem

Fast iterative algorithms:

ADMM/Split-Bregman,

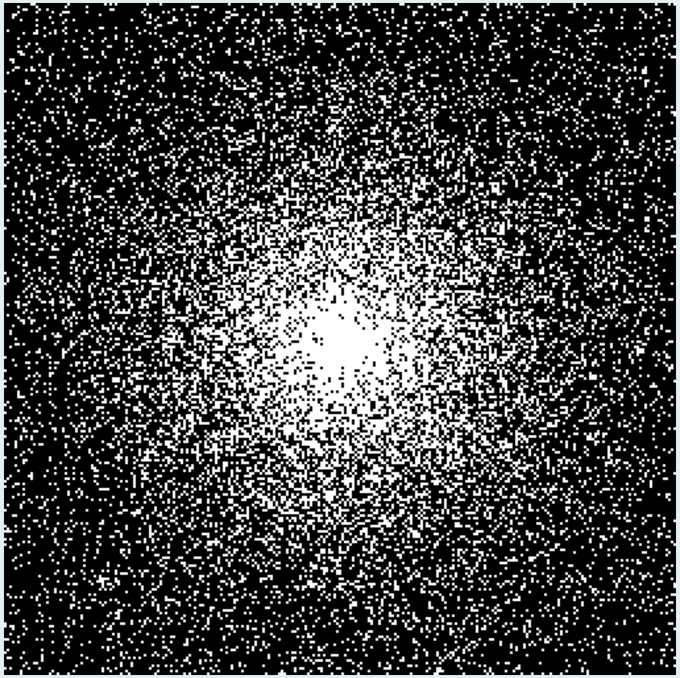
FISTA, Primal-Dual, etc.

$\Omega =$



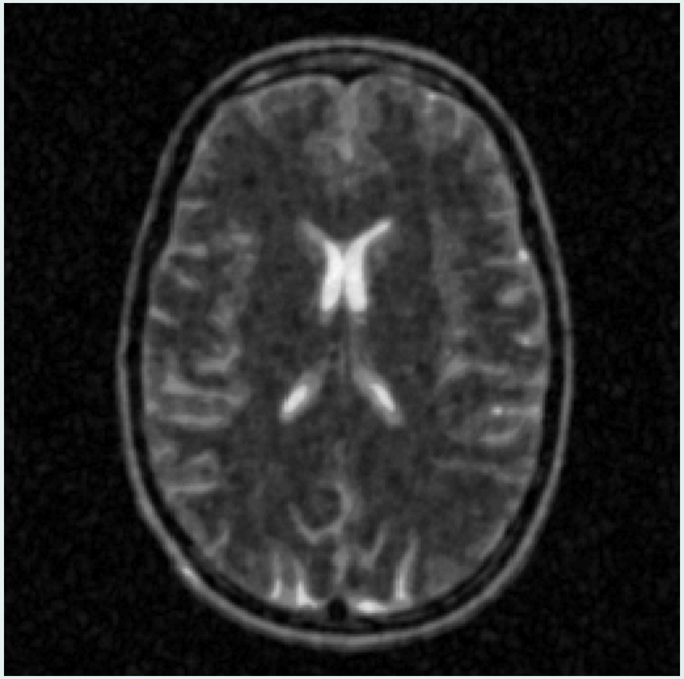
Sample locations

Example:



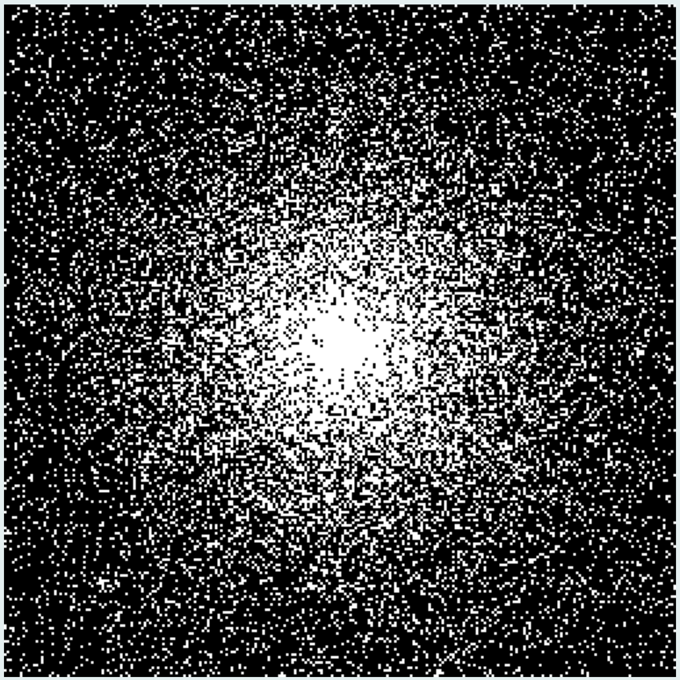
25% Random
Fourier samples
(variable density)

$\xrightarrow{\mathbf{DFT}^{-1}}$



Rel. Error = 30%

Example:



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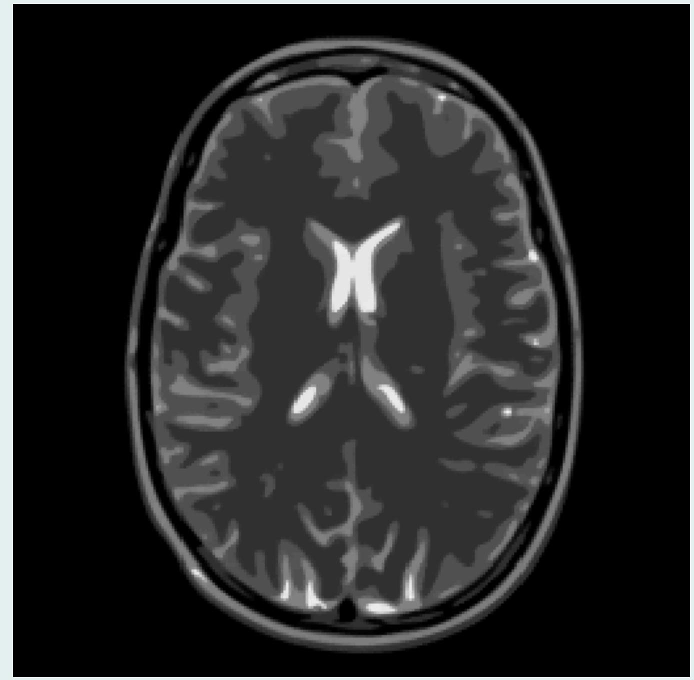
TV-min



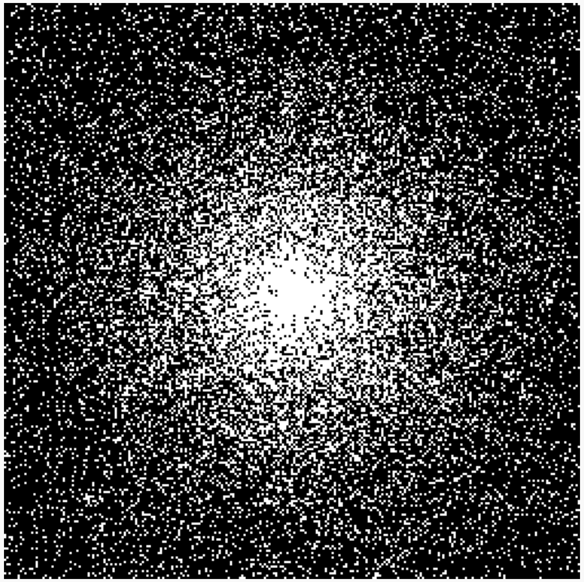
Rel. Error = 5%

Theorem [Krahmer & Ward, 2012]:

If $\mathbf{f} \in \mathbb{C}^{N \times N}$ has ***s*-sparse gradient**, then **f** is the unique solution to (TV-min) with high probability provided the number of random* Fourier samples **m** satisfies

$$\mathbf{m} \gtrsim \mathbf{s} \log^3(\mathbf{s}) \log^5(\mathbf{N})$$

* Variable density sampling



Summary of
DISCRETE PARADIGM

- Approximate $\mathcal{F} \rightarrow \mathbf{DFT}$

- **Fully sampled:**
  Fast reconstruction by $\mathbf{DFT}^{-1}$

- **Under-sampled (*Compressed sensing*):**
  Exploit sparse models & convex optimization
  – E.g. TV-minimization
  – Recovery guarantees

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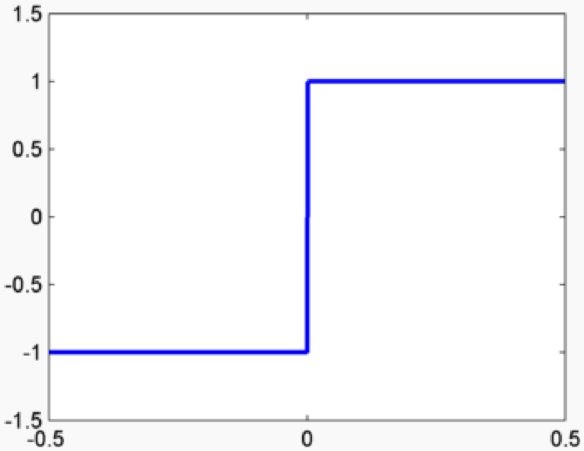
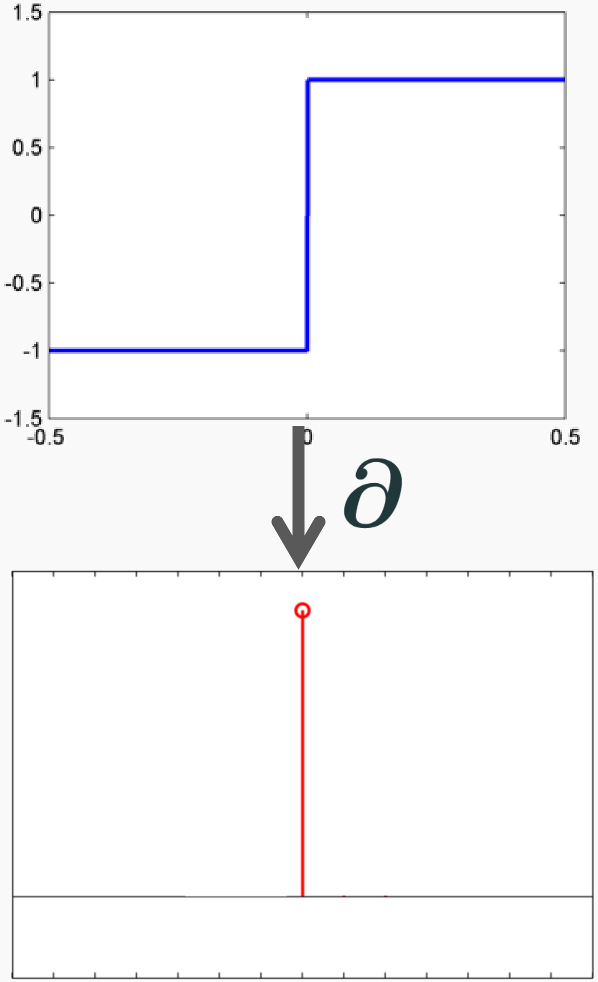
Problem: The DFT Destroys Sparsity!

Continuous



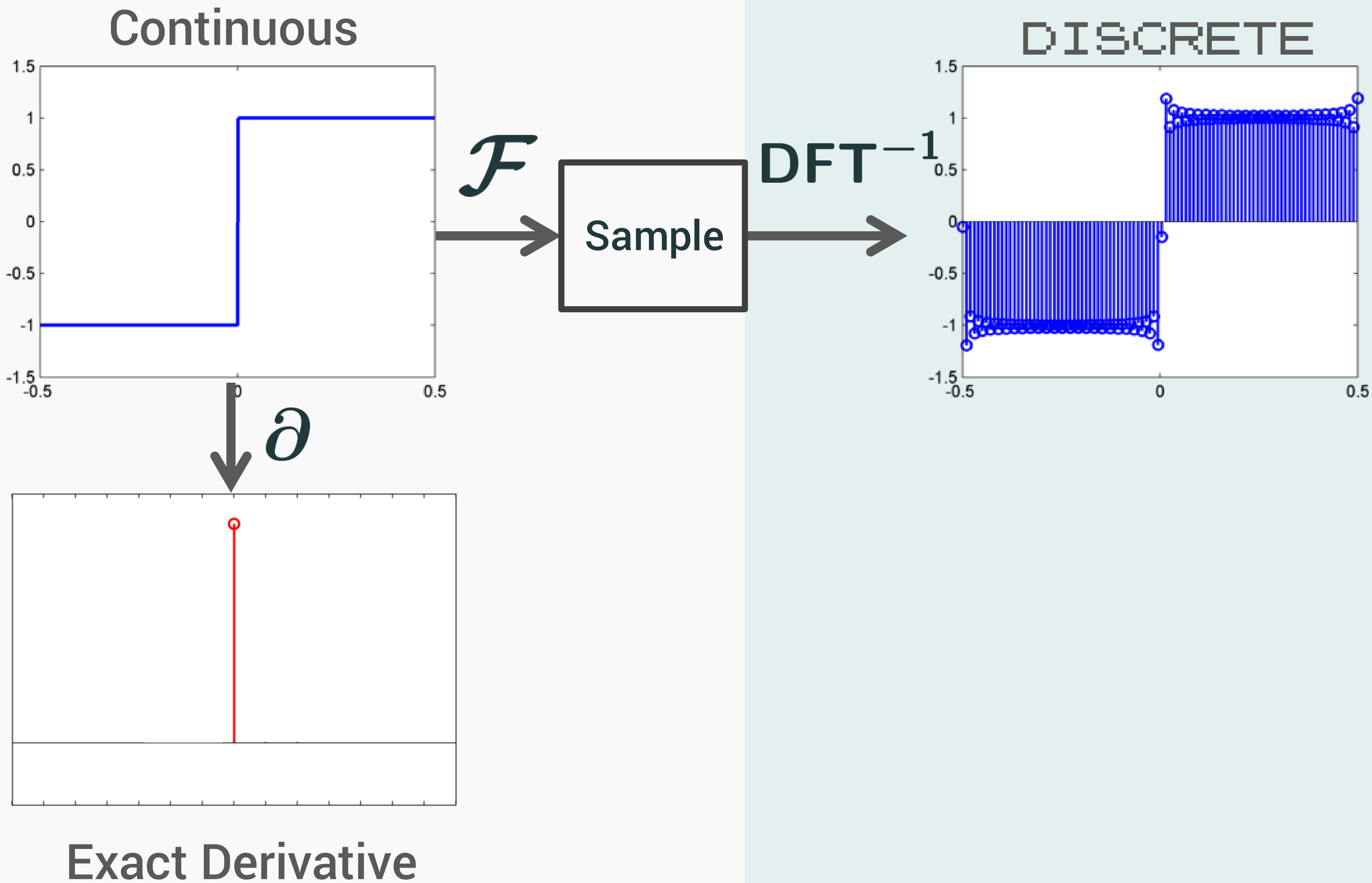
Problem: The DFT Destroys Sparsity!

Continuous



Exact Derivative

Problem: The DFT Destroys Sparsity!



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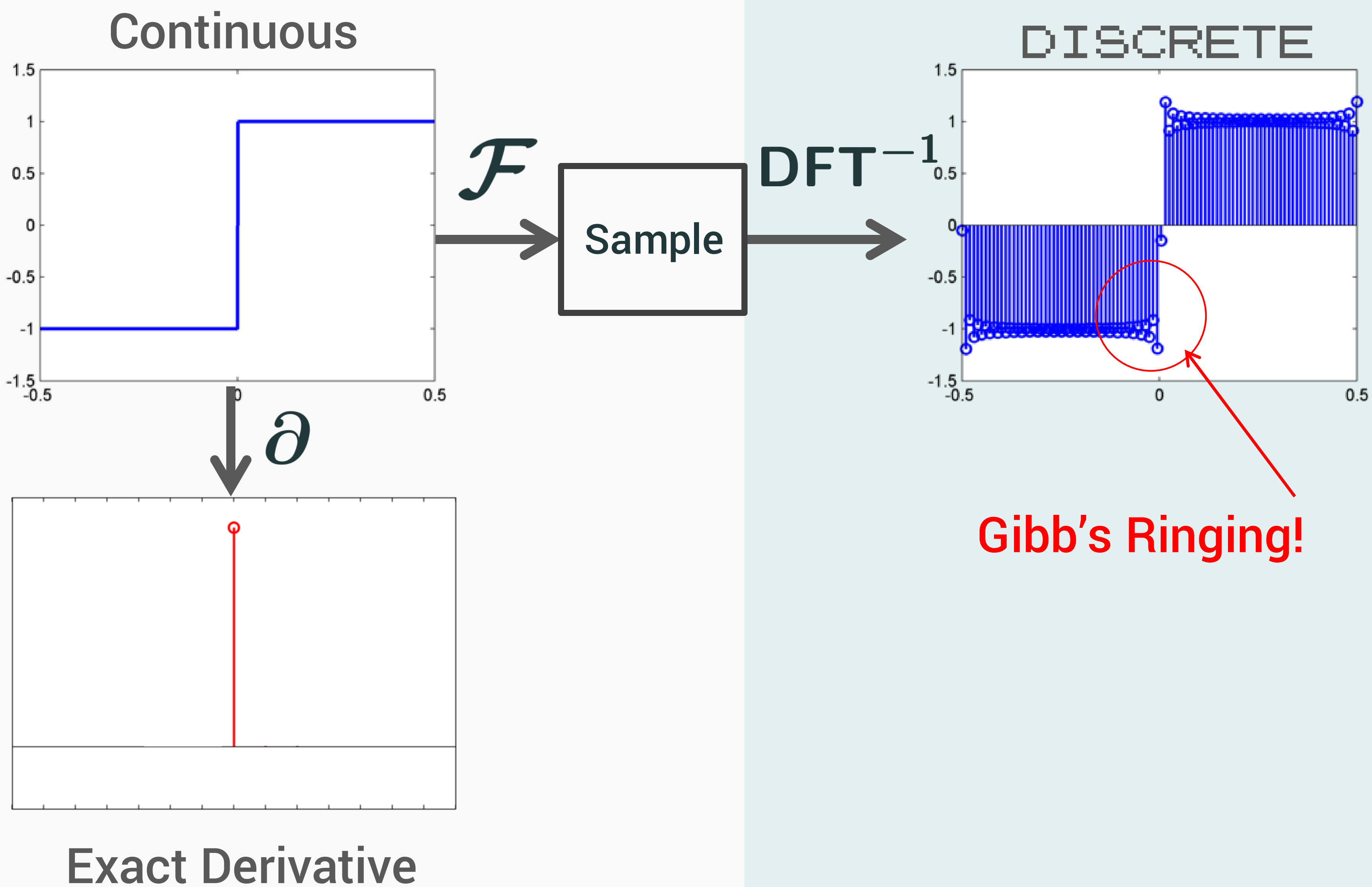
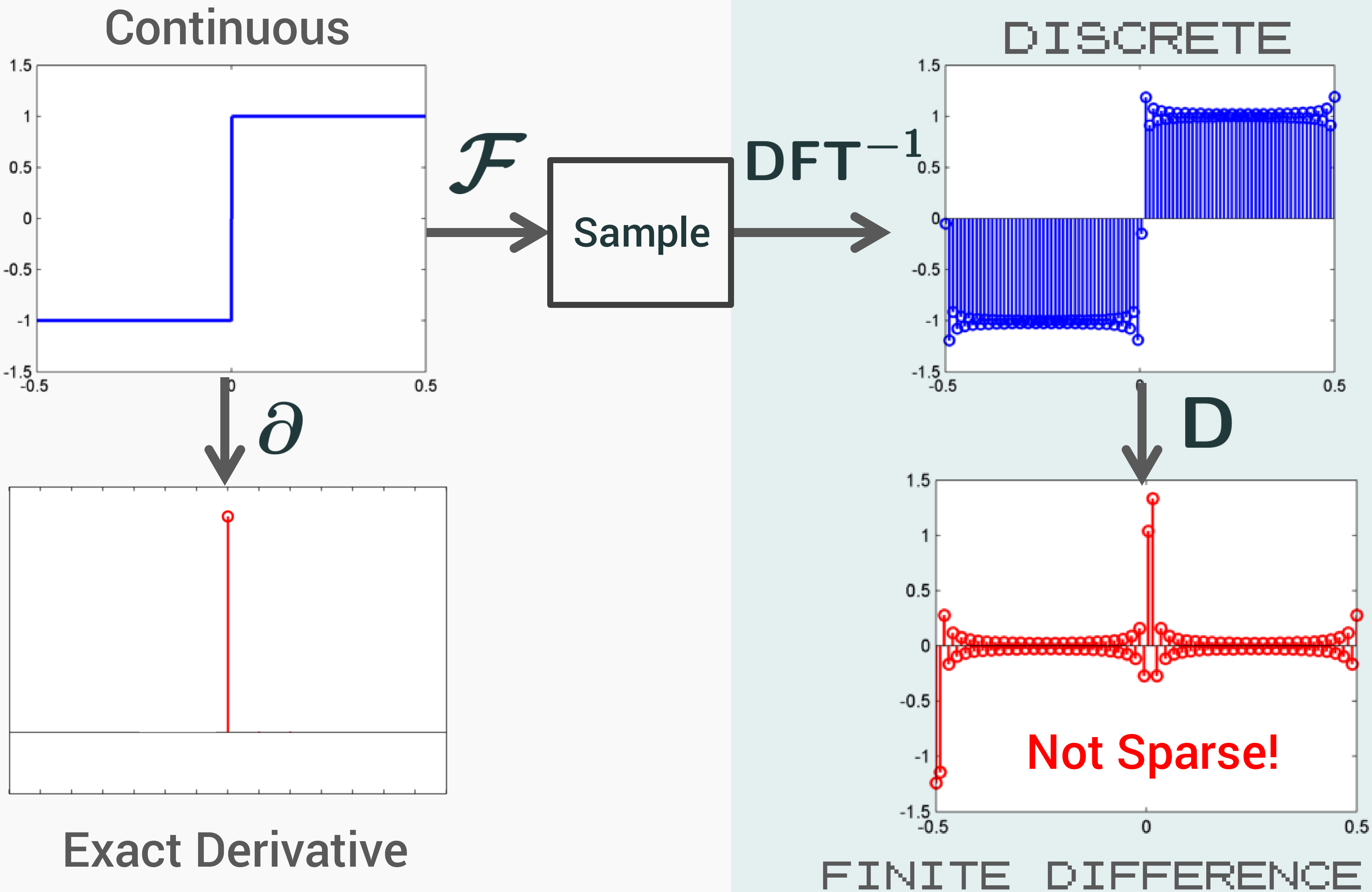


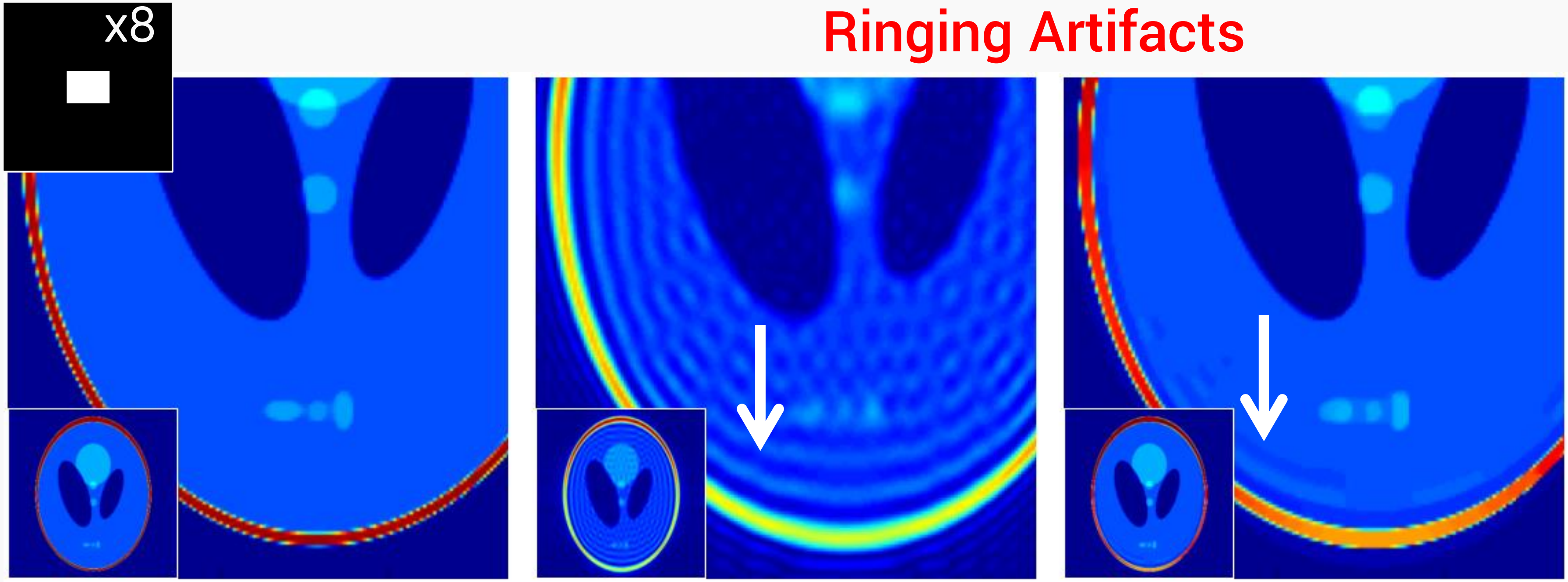
Problem: The DFT Destroys Sparsity!



Consequence: TV fails in super-resolution setting

Fourier

Ringing Artifacts

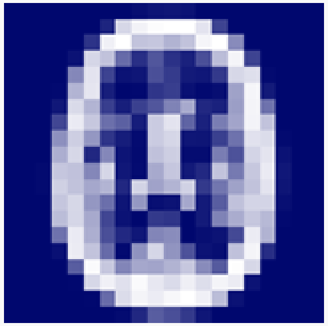


(a) Fully sampled (b) IFFT, SNR=10.8dB (c) TV, SNR=16.6dB

Can we move beyond the DISCRETE PARADIGM in Compressive Imaging?

Challenges:

- Continuous domain sparsity $\neq$ Discrete domain sparsity



- What are the continuous domain analogs of sparsity?
- Can we pose recovery as a convex optimization problem?
- Can we give recovery guarantees, *a la* TV-minimization?

New *Off-the-Grid* Imaging Framework: *Theory*

Classical Off-the-Grid Method: Prony (1795)

Uniform time samples

Off-the-grid frequencies

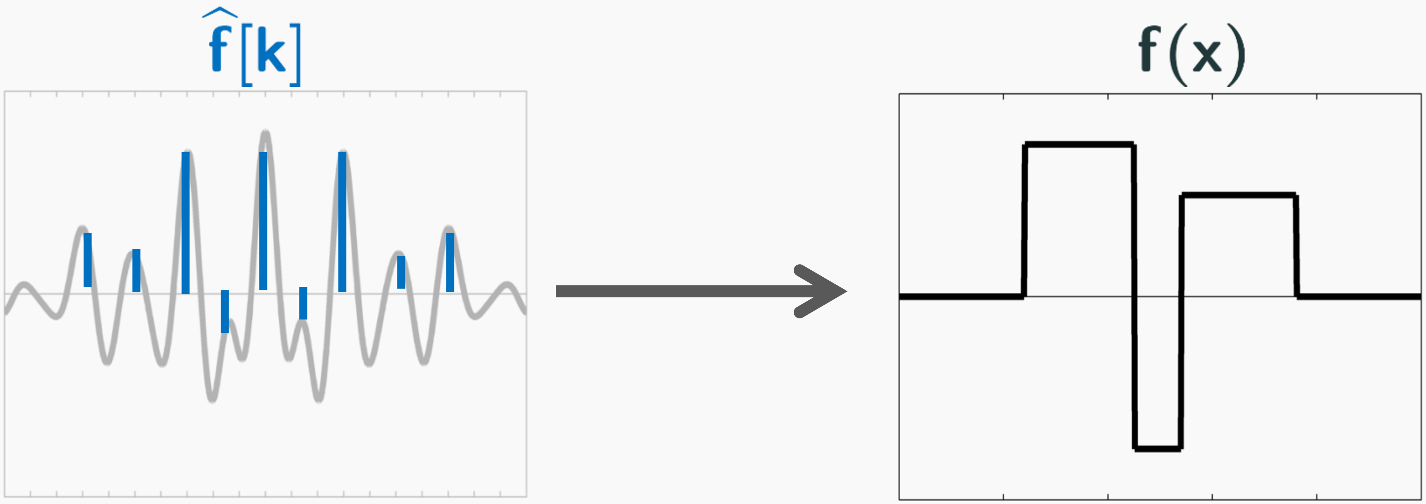
- Robust variants:

  Pisarenko (1973), MUSIC (1986), ESPRIT (1989),

  Matrix pencil (1990) . . . Atomic norm (2011)

Main inspiration: Finite-Rate-of-Innovation (FRI)

[Vetterli et al., 2002]

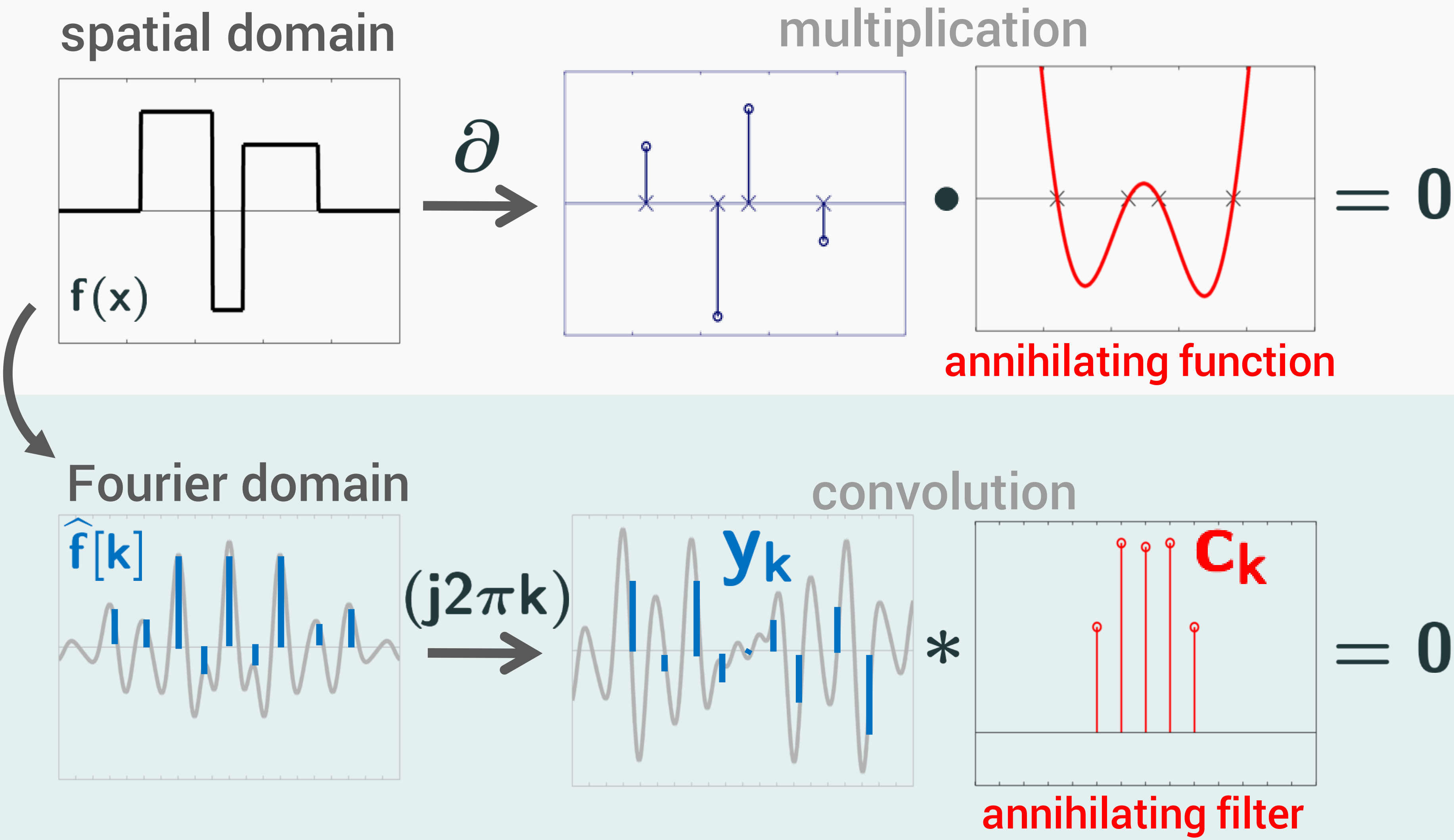


Uniform
Fourier samples

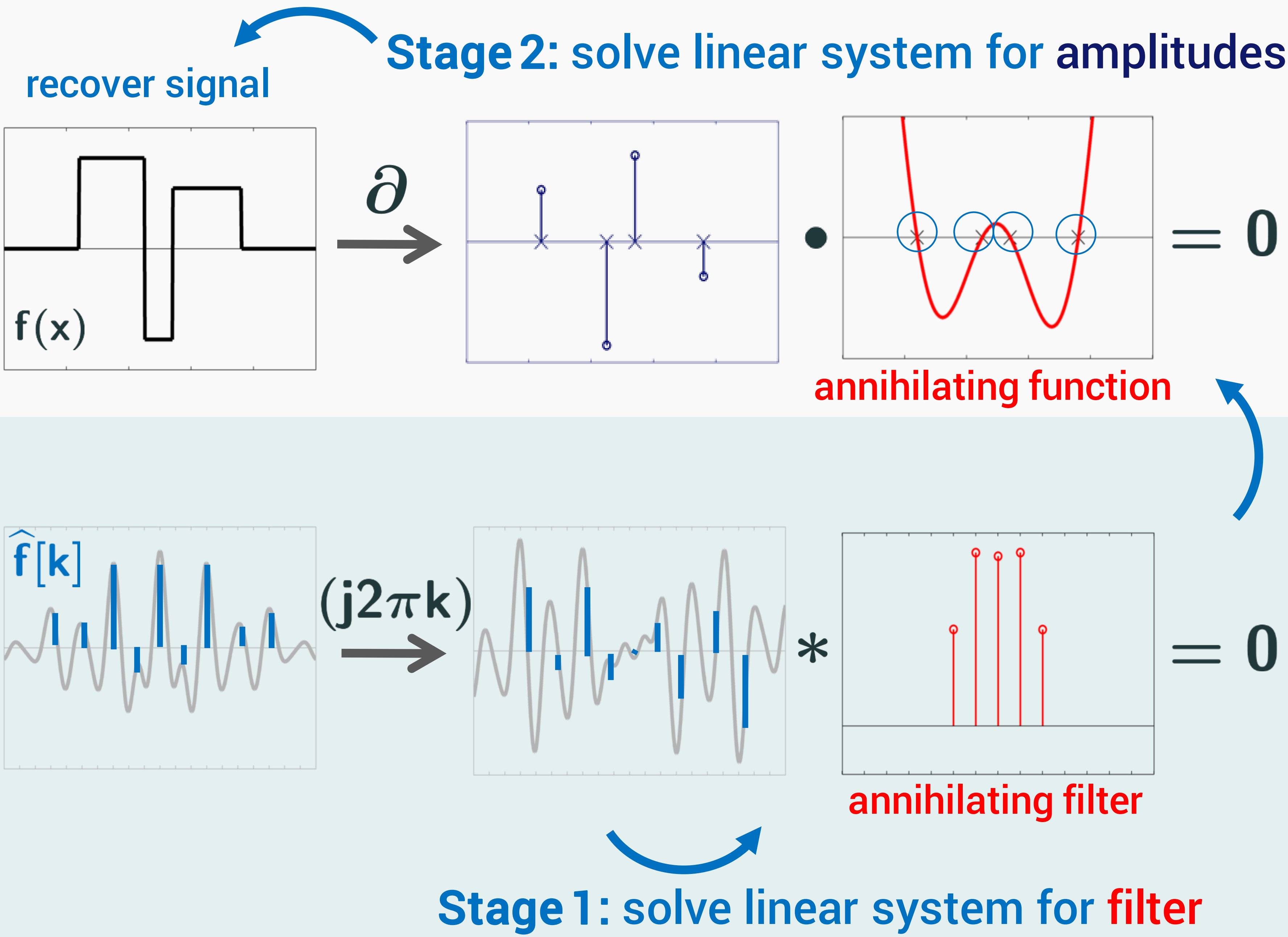
Off-the-grid
PWC signal

- Recent extension to 2-D images:

  Pan, Blu, & Dragotti (2014), "Sampling Curves with FRI".

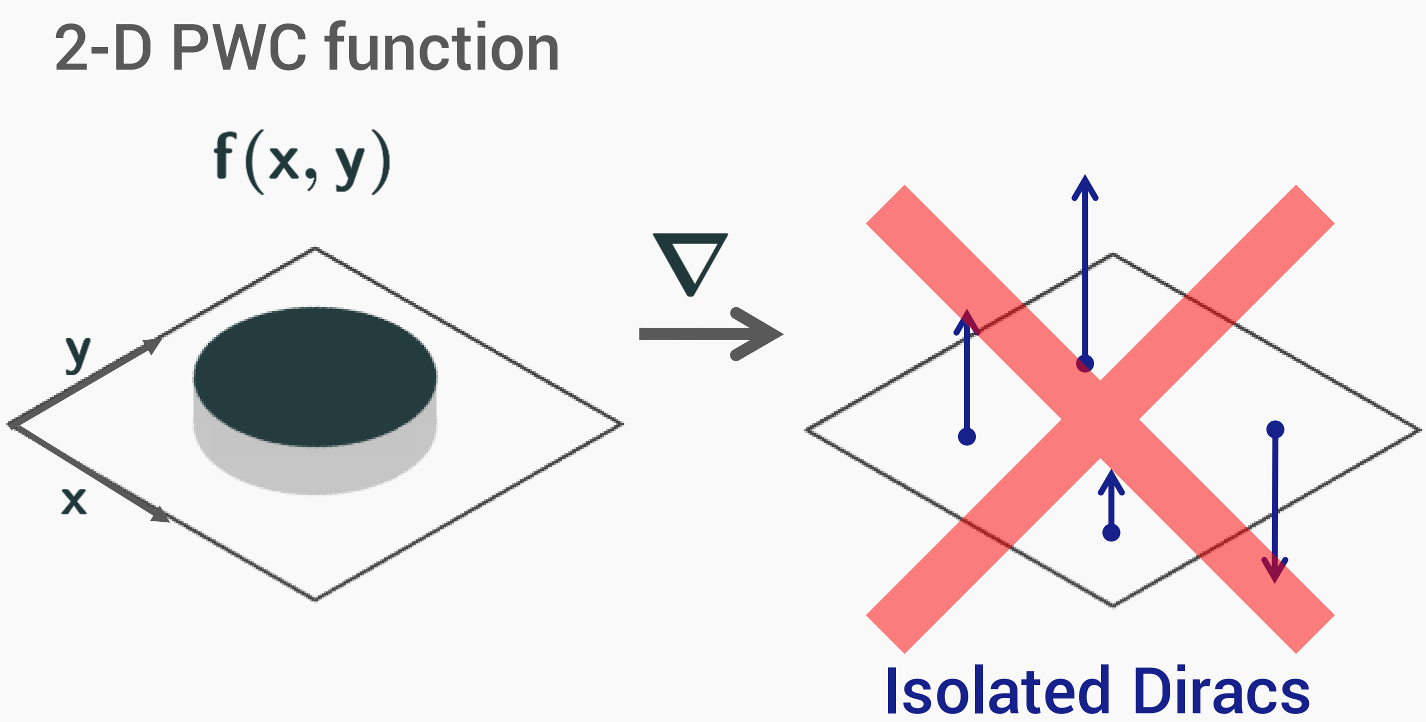


Annihilation Relation: $\sum_k y_{\ell-k} c_k = 0$



Stage 2: solve linear system for amplitudes
recover signal
f(x)
∂
annihilating function
= 0
f̂[k]
(j2πk)
*
annihilating filter
= 0
Stage 1: solve linear system for filter

Challenges extending FRI to higher dimensions:
Singularities not isolated



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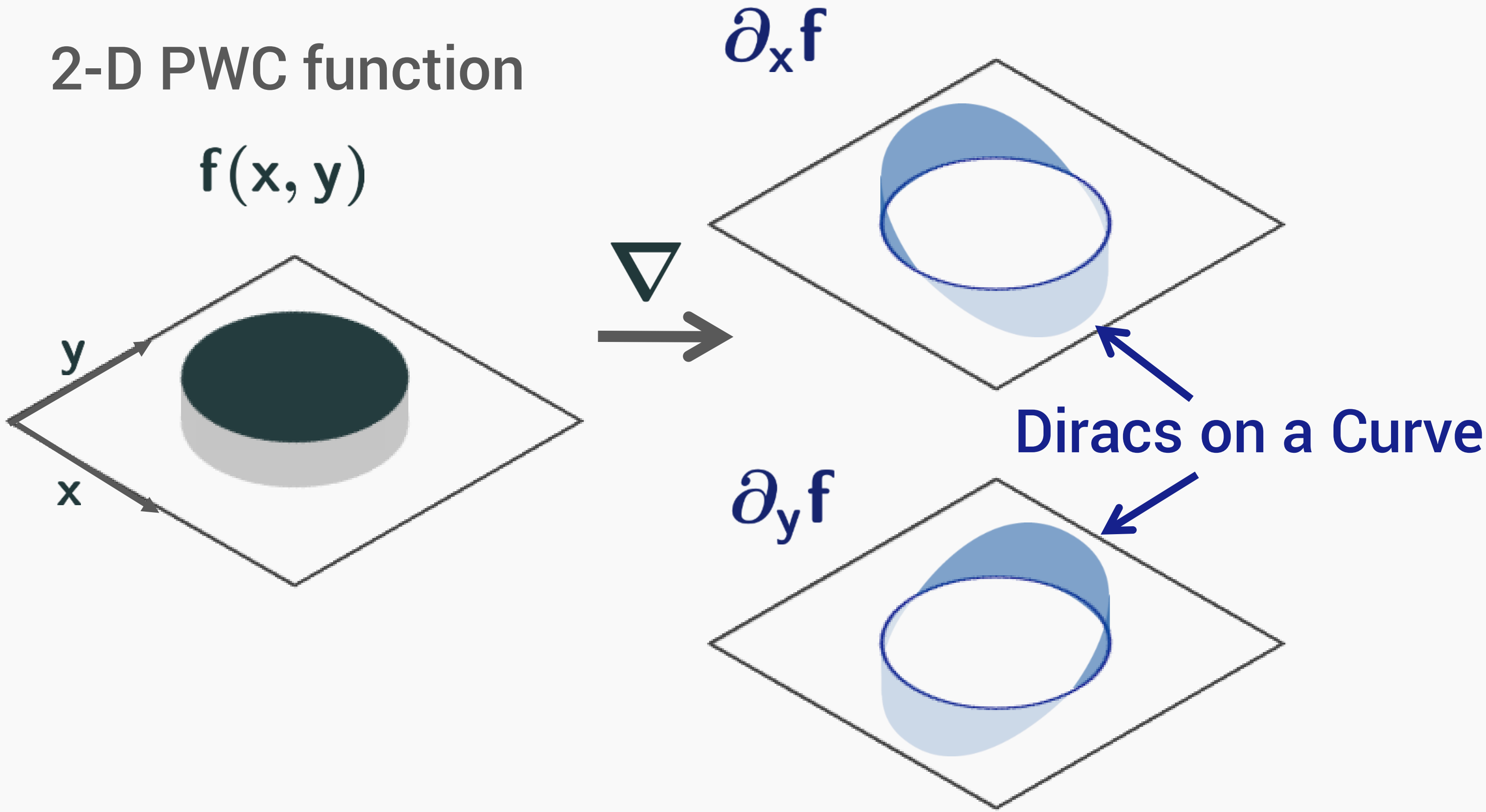
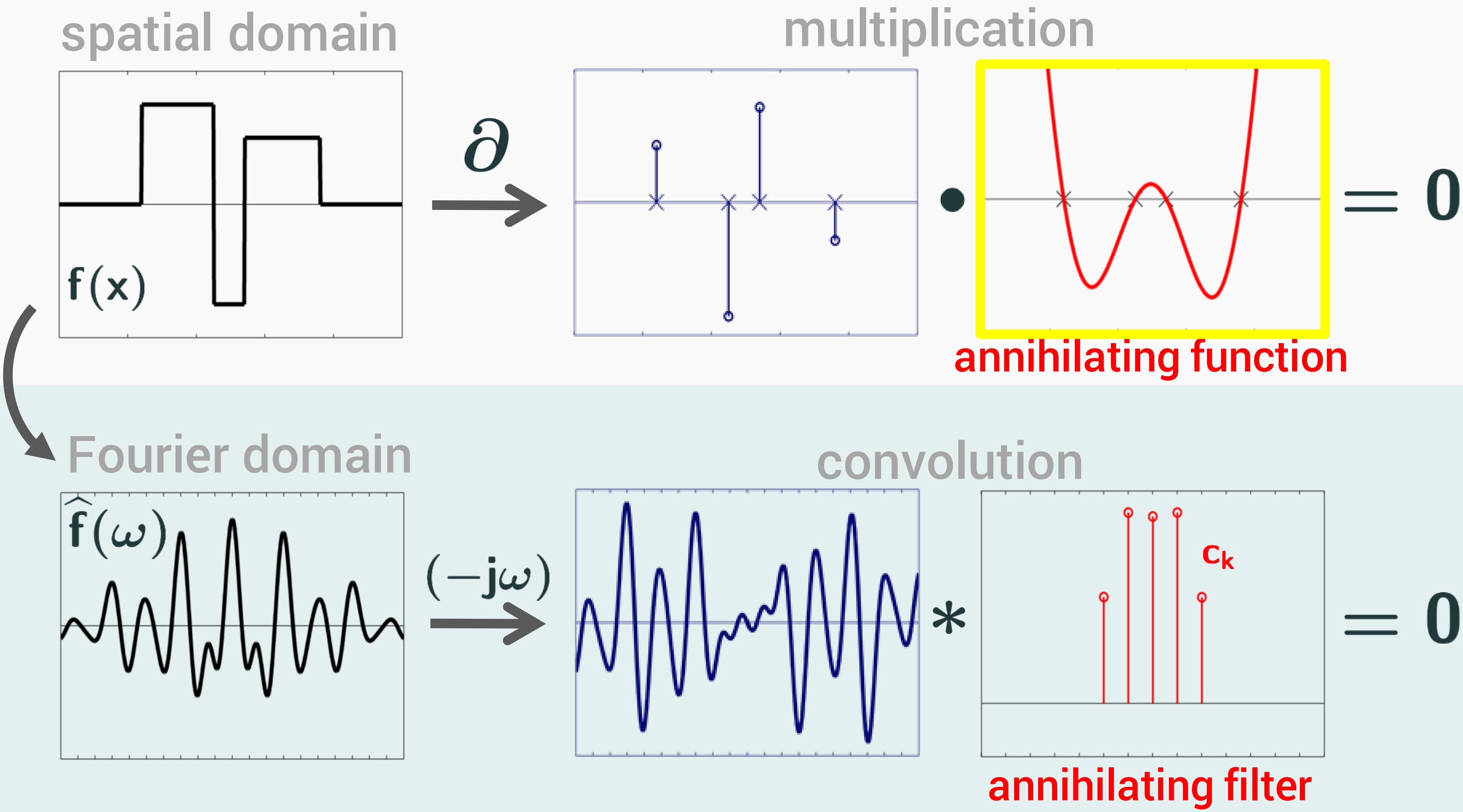
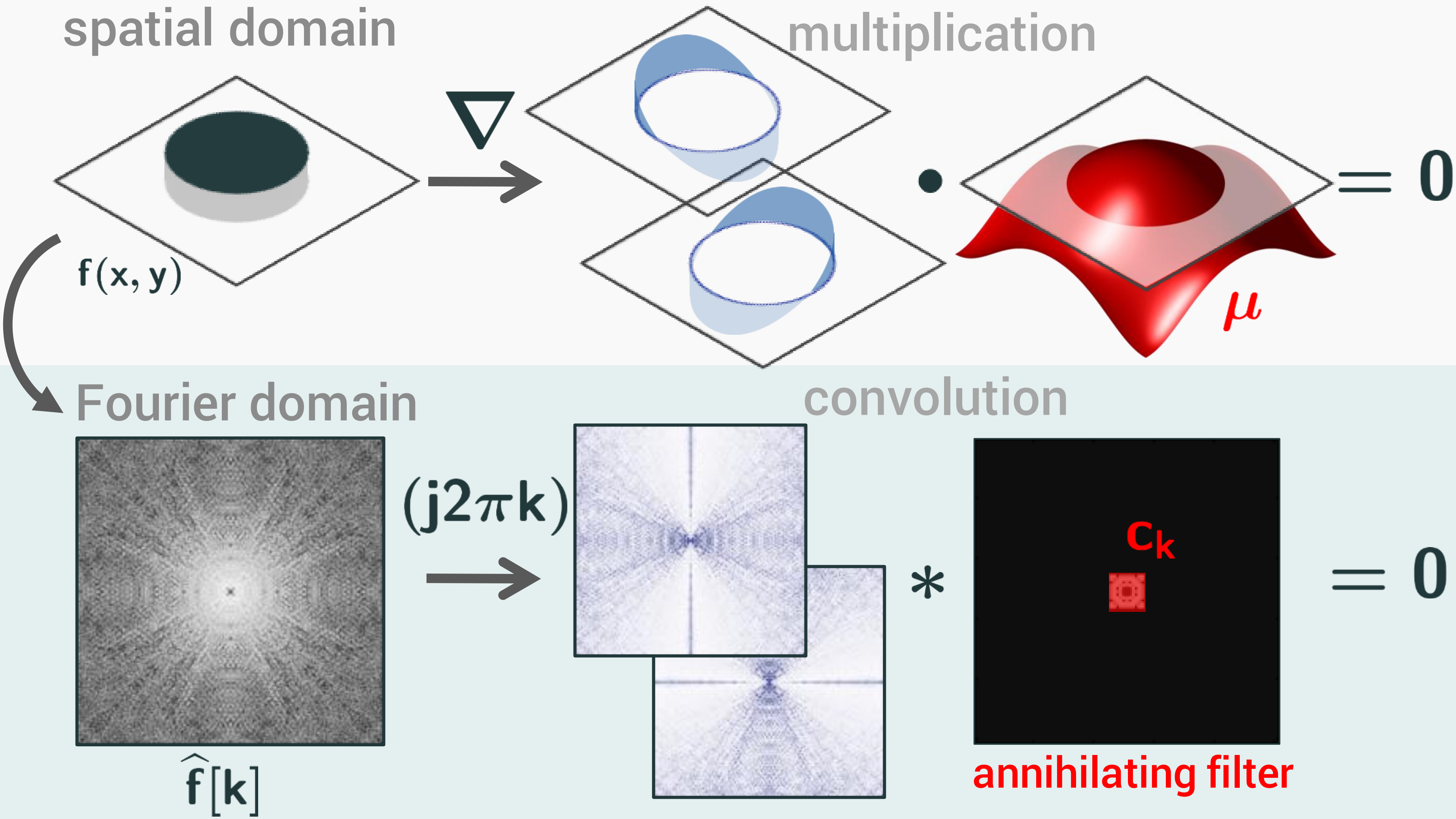


Recall 1-D Case...

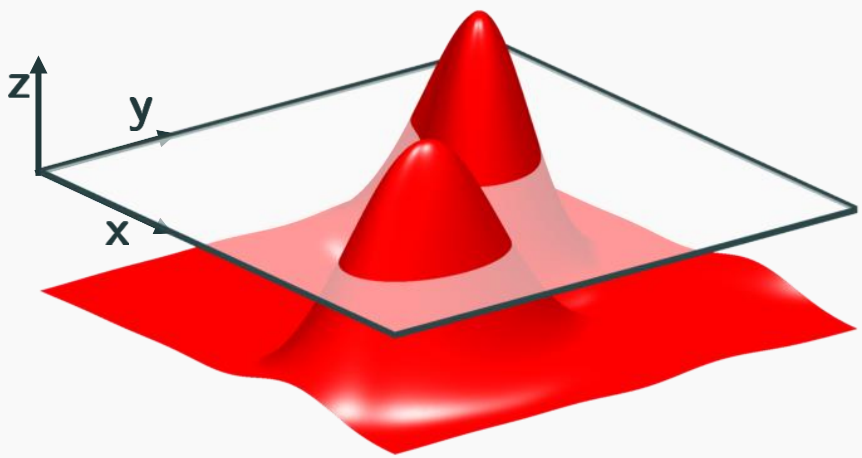


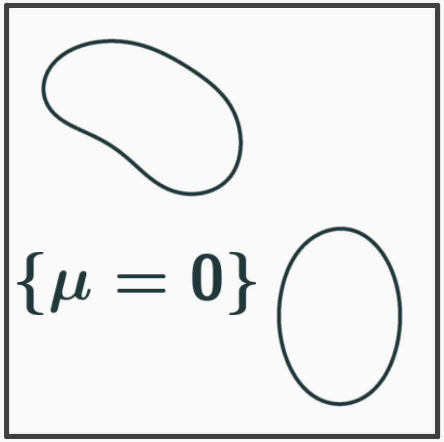
2-D PWC functions satisfy an annihilation relation



Annihilation relation: $\sum_k \widehat{\nabla f}[\ell - k] c_k = 0$

Can recover edge set when it is the
zero-set of a 2-D trigonometric polynomial
[Pan et al., 2014]



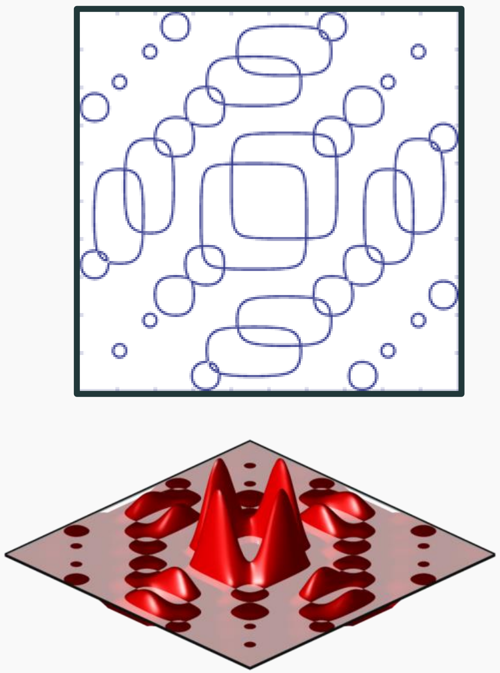


$$\mu(x, y) = \sum_{(k,l)\in\Lambda} c_{k,l}\, e^{j2\pi(kx+ly)}$$

"FRI Curve"

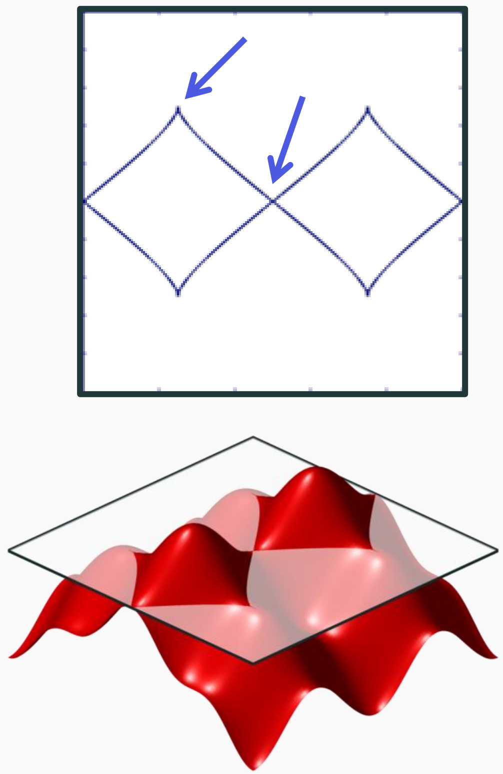
FRI curves can represent complicated edge geometries with few coefficients

Multiple curves & intersections



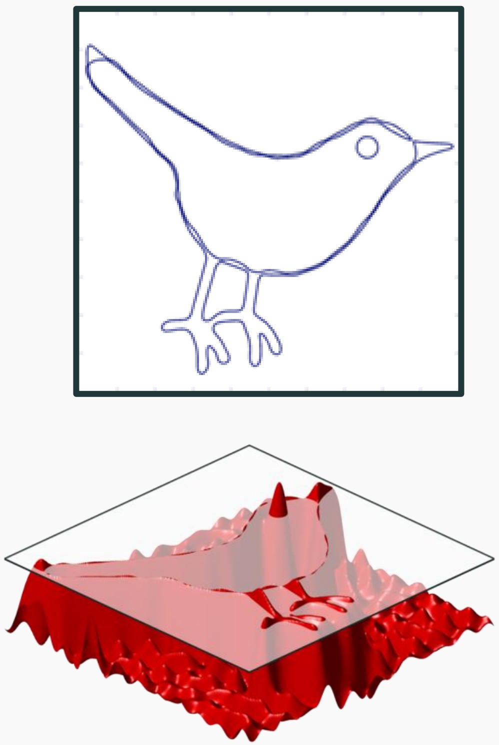
13x13 coefficients

Non-smooth points



7x9 coefficients

Approximate arbitrary curves



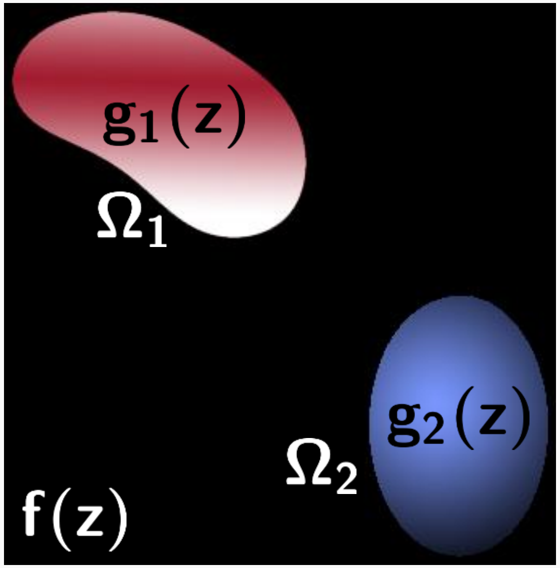
25x25 coefficients

We give an improved theoretical framework for higher dimensional FRI recovery

- *[Pan et al., 2014]* derived annihilation relation for piecewise complex analytic signal model

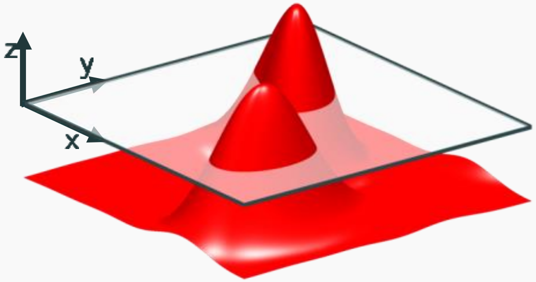
$$f(z) = \sum_{i=1}^{N} g_i(z) \cdot 1_{\Omega_i}(z)$$

$$\text{s.t.} \quad g_i \text{ analytic in } \Omega_i$$



- Not suitable for natural images
- 2-D only
- Recovery is ill-posed:

  Infinite DoF

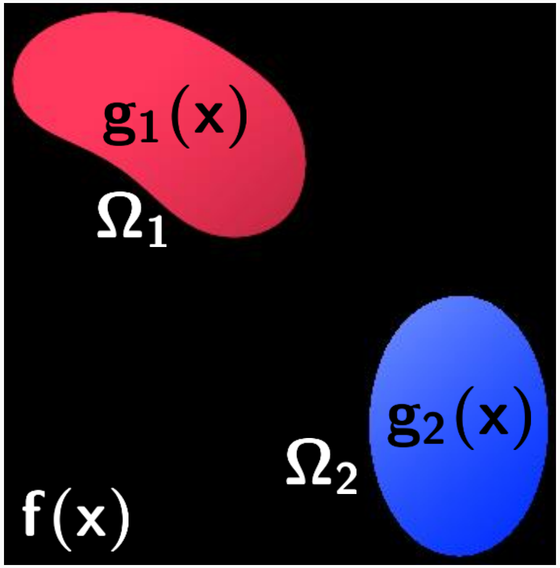


We give an improved theoretical framework for higher dimensional FRI recovery *[O. & Jacob, SampTA 2015]*

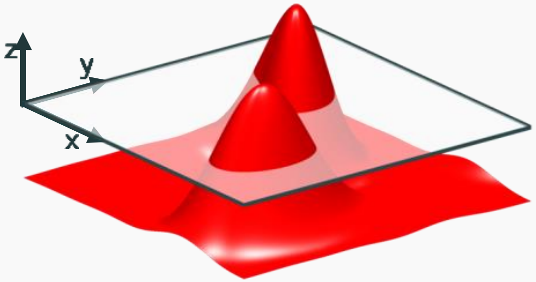
- Proposed model: piecewise smooth signals

$$f(x) = \sum_{i=1}^{N} g_i(x) \cdot 1_{\Omega_i}(x)$$

$$\text{s.t.} \quad g_i \text{ smooth in } \Omega_i$$



- Extends easily to n-D
- Provable sampling guarantees
- Fewer samples necessary for recovery

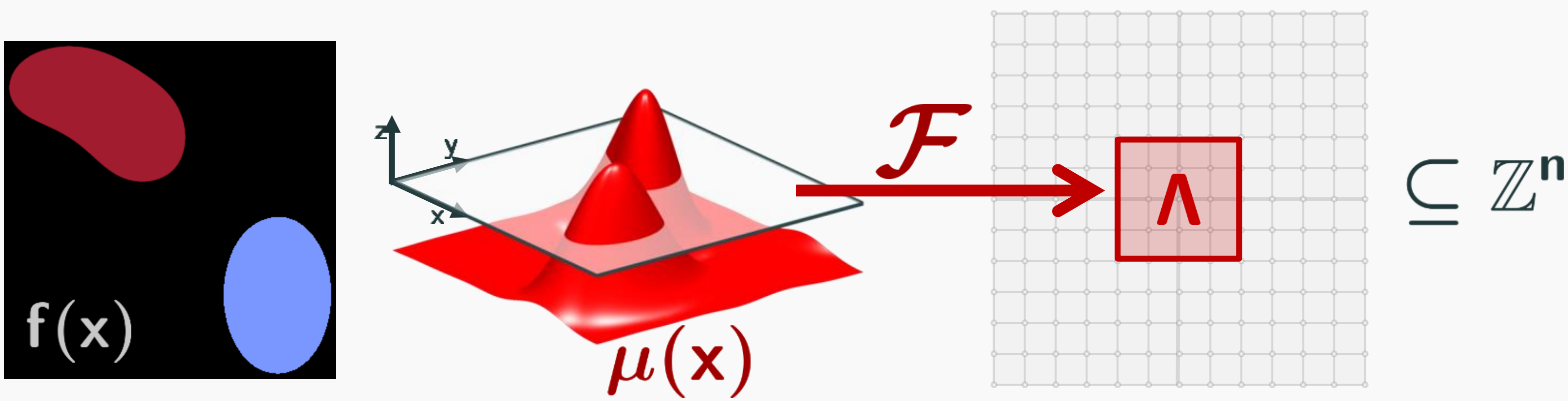


Annhilation relation for PWC signals

Prop: If *f* is PWC with edge set $\mathbf{E} \subseteq \{\mu = 0\}$ for μ bandlimited to $\mathbf{\Lambda}$ then

$$\sum_{\mathbf{k}\in\mathbf{\Lambda}} \widehat{\mu}[\mathbf{k}]\widehat{\partial \mathbf{f}}[\ell - \mathbf{k}] = \mathbf{0}, \quad \forall \ell \in \mathbb{Z}^{\mathbf{n}}$$

any 1st order partial derivative



Annhilation relation for PWC signals

Prop: If *f* is PWC with edge set $\mathbf{E} \subseteq \{\mu = \mathbf{0}\}$

for μ bandlimited to $\mathbf{\Lambda}$ then

$$\sum_{\mathbf{k}\in\mathbf{\Lambda}} \widehat{\mu}[\mathbf{k}]\widehat{\partial\mathbf{f}}[\ell - \mathbf{k}] = \mathbf{0}, \quad \forall \ell \in \mathbb{Z}^{\mathbf{n}}$$

any 1st order partial derivative

Proof idea:

Show $\mu \cdot \partial\mathbf{f} = \mathbf{0}$ as tempered distributions

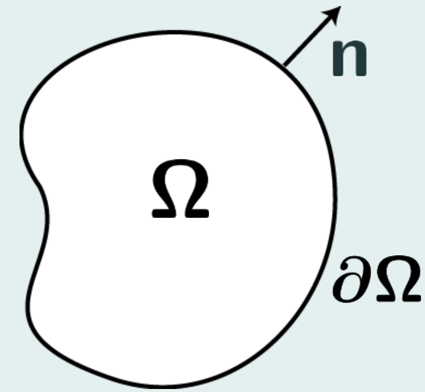
Use convolution theorem

Distributional derivative of indicator function:

smooth test function

$$
\begin{aligned}
\langle \partial_j \mathbf{1}_\Omega, \varphi \rangle &= -\langle \mathbf{1}_\Omega, \partial_j \varphi \rangle \\
&= -\int_\Omega \partial_j \varphi \, \mathbf{dx} \\
&= -\oint_{\partial\Omega} \varphi \, \mathbf{n_j} \, \mathbf{d}\sigma
\end{aligned}
$$

divergence theorem



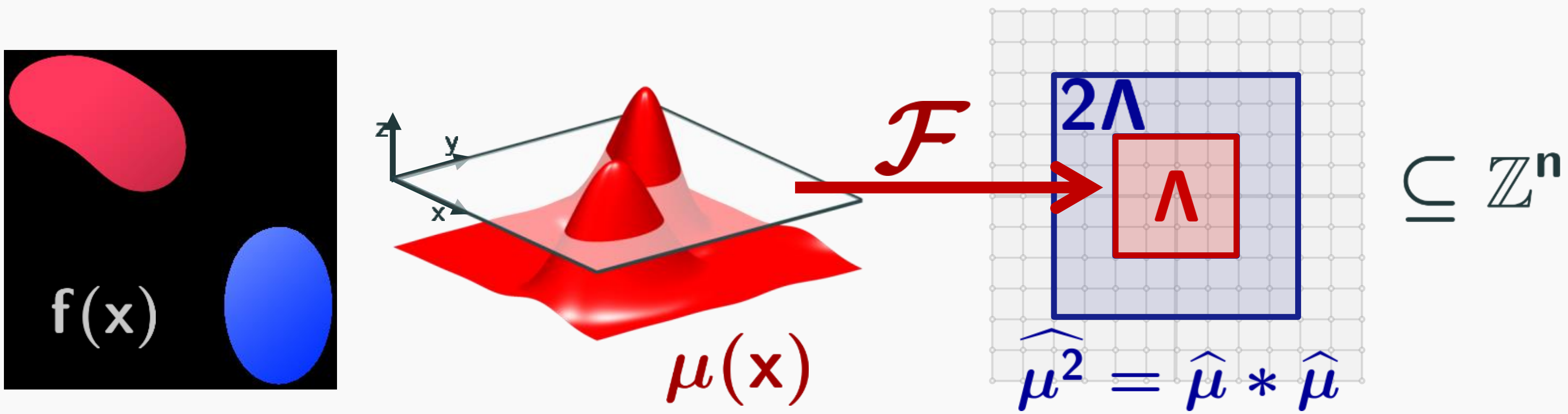
Weighted curve integral

Annhilation relation for PW linear signals

Prop: If **f** is PW linear, with edge set $\mathbf{E} \subseteq \{\mu = 0\}$ and μ bandlimited to **Λ** then

$$\sum_{\mathbf{k} \in 2\Lambda} \widehat{\mu^2}[\mathbf{k}]\,\widehat{\partial^2 \mathbf{f}}[\ell - \mathbf{k}] = 0, \quad \forall \ell \in \mathbb{Z}^{\mathbf{n}}$$

any 2nd order partial derivative



Annhilation relation for PW linear signals

Prop: If **f** is PW linear, with edge set $\mathbf{E} \subseteq \{\mu = 0\}$

and μ bandlimited to **Λ** then

$$\sum_{\mathbf{k}\in 2\boldsymbol{\Lambda}} \widehat{\mu^2}[\mathbf{k}]\,\widehat{\partial^2 \mathbf{f}}[\ell - \mathbf{k}] = \mathbf{0}, \quad \forall \ell \in \mathbb{Z}^{\mathbf{n}}$$

any 2nd order partial derivative

Proof idea: $\mathbf{f} = \mathbf{g} \cdot 1_\Omega, \quad \mathbf{g}$ **linear**

product rule x2 $\partial^2 \mathbf{f} = \cancel{\partial^2 \mathbf{g} \cdot 1_\Omega} + 2\partial \mathbf{g} \cdot \partial 1_\Omega + \mathbf{g} \cdot \partial^2 1_\Omega$

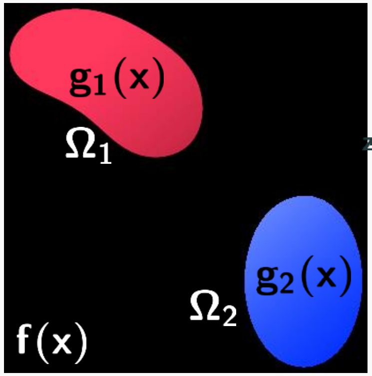
annihilated by μ^2

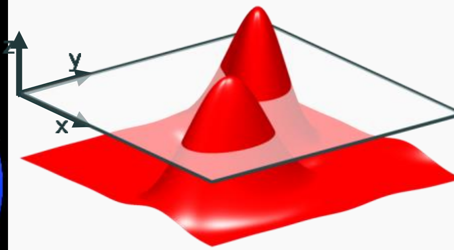
Can extend annihilation relation to a wide class of piecewise smooth images.

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{g}_{\mathbf{i}}(\mathbf{x}) \cdot 1_{\Omega_{\mathbf{i}}}(\mathbf{x})$$

$$\text{s.t.} \quad \mathbf{D}\mathbf{g}_{\mathbf{i}} = \mathbf{0} \text{ in } \Omega_{\mathbf{i}}$$

Any constant coeff. differential operator

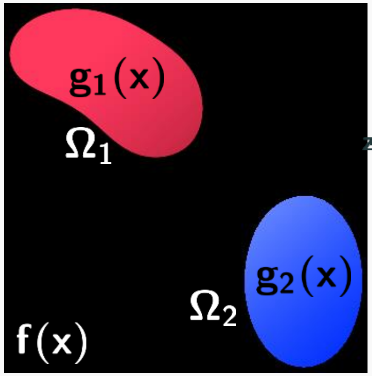


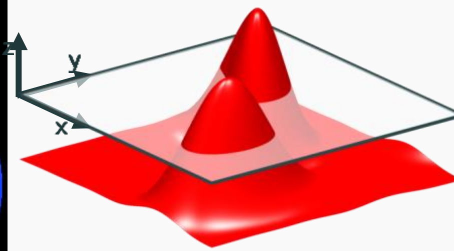


Can extend annihilation relation to a wide class of piecewise smooth images.

$$f(x) = \sum_{i=1}^{N} g_i(x) \cdot 1_{\Omega_i}(x)$$

$$\text{s.t.} \quad D g_i = 0 \text{ in } \Omega_i$$





Signal Model:	Choice of Diff. Op.:	
PW Constant	$D = \nabla$	1^{st} order
PW Analytic*	$D = \partial_x + j\partial_y$	

Can extend annihilation relation to a wide class of **piecewise smooth** images.

$$f(x) = \sum_{i=1}^{N} g_i(x) \cdot 1_{\Omega_i}(x)$$

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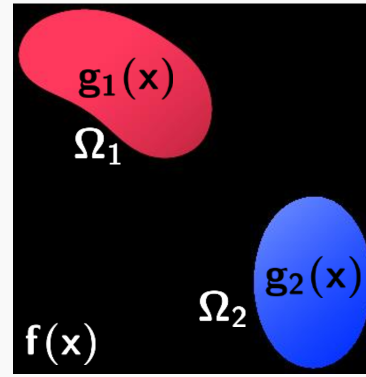

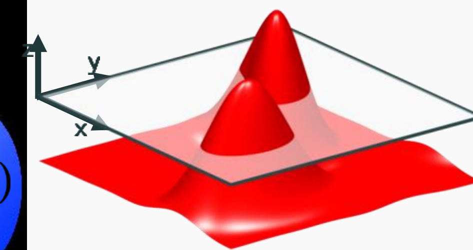

Signal Model:	Choice of Diff. Op.:	
PW Constant	$D = \nabla$	1st order
PW Analytic*	$D = \partial_x + j\partial_y$	1st order
PW Harmonic	$D = \Delta$	2nd order
PW Linear	$D = \{\partial_{xx}, \partial_{xy}, \partial_{yy}\}$	2nd order

Can extend annihilation relation to a wide class of **piecewise smooth** images.

$$f(x) = \sum_{i=1}^{N} g_i(x) \cdot 1_{\Omega_i}(x)$$

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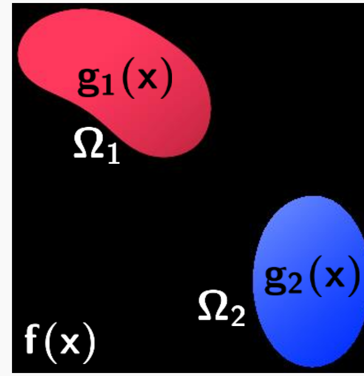
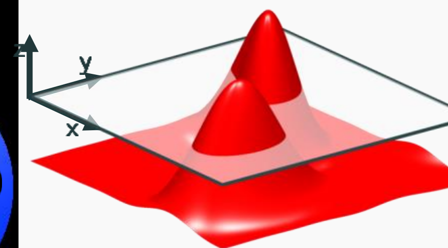




Signal Model:	Choice of Diff. Op.:	
PW Constant	$D = \nabla$	1st order
PW Analytic*	$D = \partial_x + j\partial_y$	
PW Harmonic	$D = \Delta$	2nd order
PW Linear	$D = \{\partial_{xx}, \partial_{xy}, \partial_{yy}\}$	
PW Polynomial	$D = \{\partial^\alpha\}_{\lvert\alpha\rvert=n}$	nth order

Sampling theorems:

Necessary and sufficient number of Fourier samples for

1. Unique recovery of edge set/annihilating polynomial
2. Unique recovery of full signal given edge set
   - Not possible for PW analytic, PW harmonic, etc.
   - Prefer PW polynomial models

➔ Focus on 2-D PW constant signals

Challenges to proving uniqueness

1-D FRI Sampling Theorem [Vetterli et al., 2002]:

A continuous-time PWC signal with K jumps can be uniquely recovered from 2K+1 uniform Fourier samples.

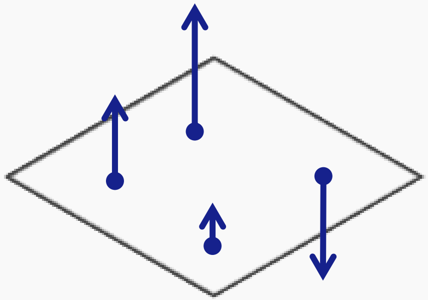
Proof (a la Prony's Method):

Form Toeplitz matrix $\mathbf{T}$ from samples, use uniqueness of Vandermonde decomposition: $\mathbf{T} = \mathbf{V}\mathbf{D}\mathbf{V}^H$

"Caratheodory Parametrization"

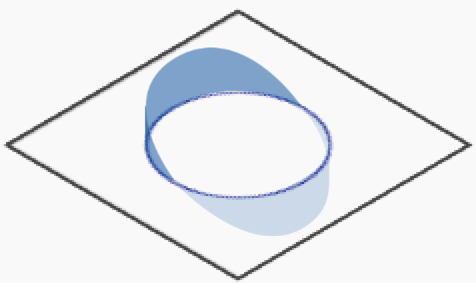
Challenges proving uniqueness, cont.

Extends to *n*-D if singularities isolated [Sidiropoulos, 2001]



$$\xrightarrow{\mathcal{F}} \quad \hat{\mathbf{f}}[\mathbf{k}] = \sum_{\mathbf{i}} \mathbf{a_i} \mathbf{e}^{-\mathbf{j}2\pi\mathbf{k}\cdot\mathbf{x_i}}$$

Not true in our case--singularities supported on curves:



$$\xrightarrow{\mathcal{F}} \quad \widehat{\nabla \mathbf{f}}[\mathbf{k}] = \oint_{\partial\Omega} \mathbf{e}^{-\mathbf{j}2\pi\mathbf{k}\cdot\mathbf{x}} \mathbf{n}\, \mathbf{ds}$$

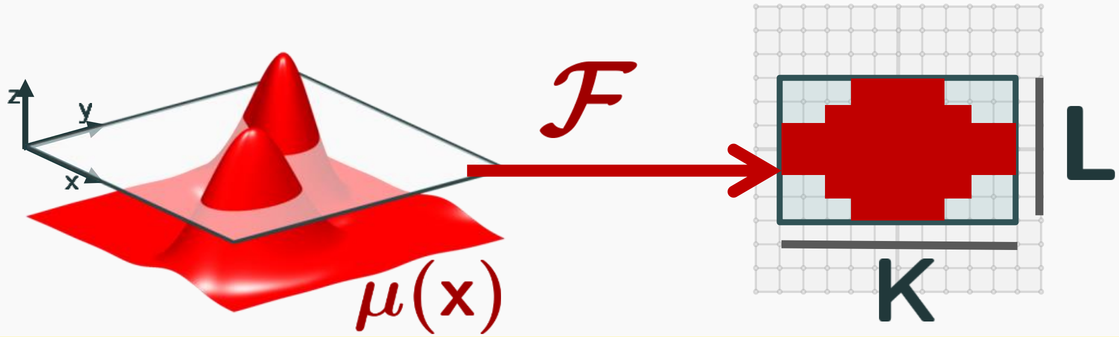
Requires new techniques:

– Spatial domain interpretation of annihilation relation

– Algebraic geometry of trigonometric polynomials

Minimal (Trigonometric) Polynomials

Define $\mathbf{deg}(\mu) = (\mathsf{K}, \mathsf{L})$ to be the dimensions of the smallest rectangle containing the Fourier support of μ



Prop: Every zero-set of a trig. polynomial **C** with no isolated points has a *unique* real-valued trig. polynomial μ_0 of minimal degree such that if $\mathsf{C} = \{\mu = 0\}$ then $\mathbf{deg}(\mu_0) \leq \mathbf{deg}(\mu)$ and $\mu = \gamma \cdot \mu_0$

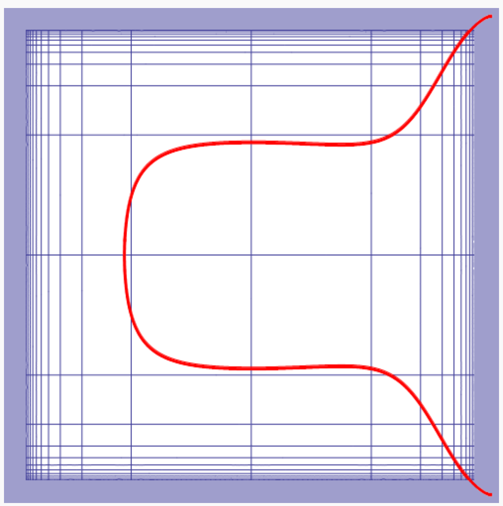
Degree of min. poly. = analog of sparsity/complexity of edge set

Proof idea: Pass to Real Algebraic Plane Curves

Zero-sets of trig polynomials of degree (K,L)

are in 1-to-1 correspondence with

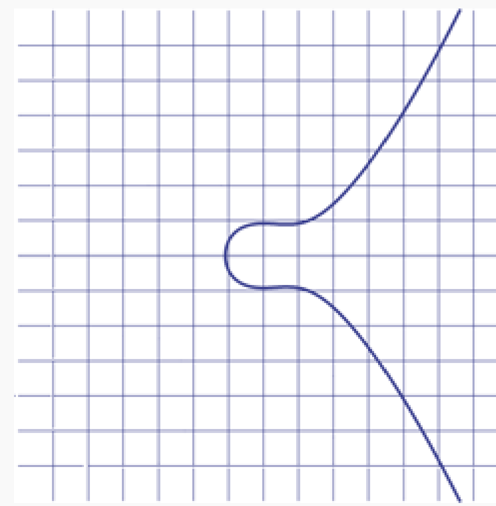
Real algebraic plane curves of degree (K,L)



$\mathbb{T}^2$

$$\mu(z, w) = 0;$$
$$|z| = |w| = 1$$

Conformal change of variables



$\mathbb{R}^2$

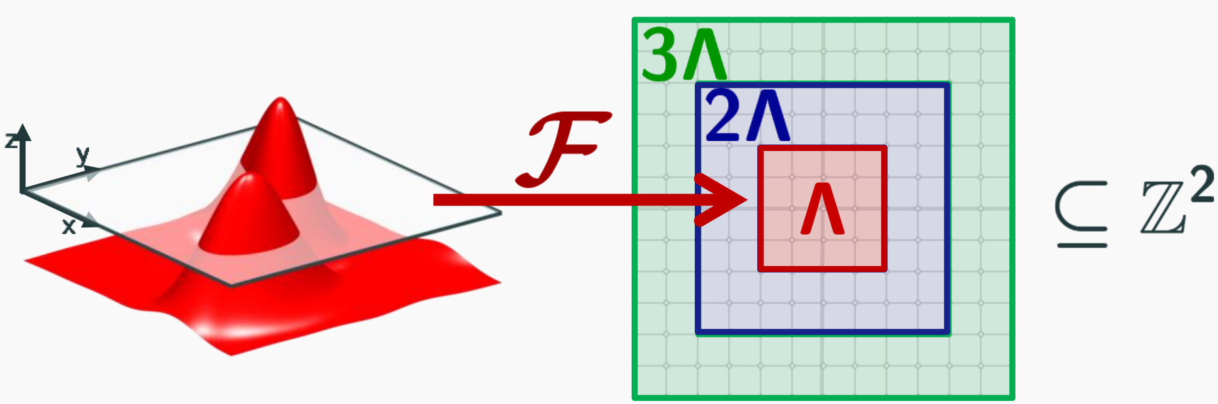
$$p(t, s) = 0;$$
$$t, s \in \mathbb{R}^2$$

Uniqueness of edge set recovery

Theorem: If $\mathbf{f}$ is PWC* with edge set $\mathbf{E} = \{\mu = 0\}$ with μ minimal and bandlimited to Λ then $\mathbf{c} = \widehat{\mu}$ is the unique solution to

$$\sum_{\mathbf{k}\in\Lambda} \mathbf{c}[\mathbf{k}]\widehat{\nabla \mathbf{f}}[\ell - \mathbf{k}] = 0 \text{ for all } \ell \in 2\Lambda$$

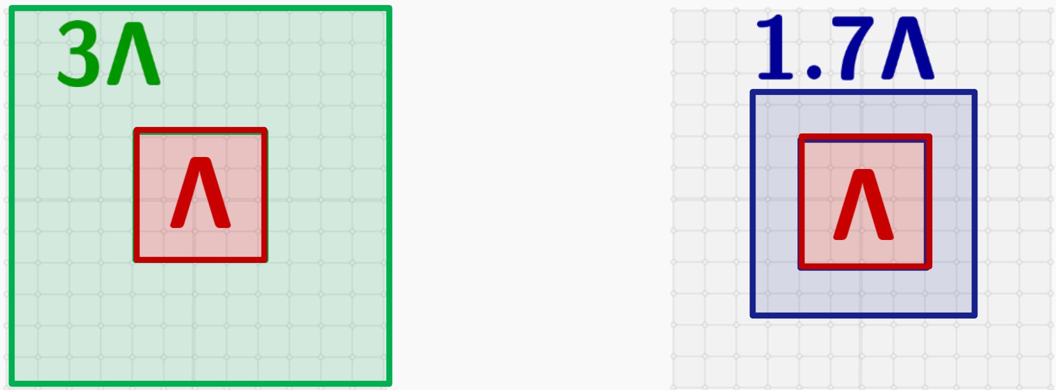
*Some geometric restrictions apply



Requires samples of $\widehat{\mathbf{f}}$ in 3Λ to build equations

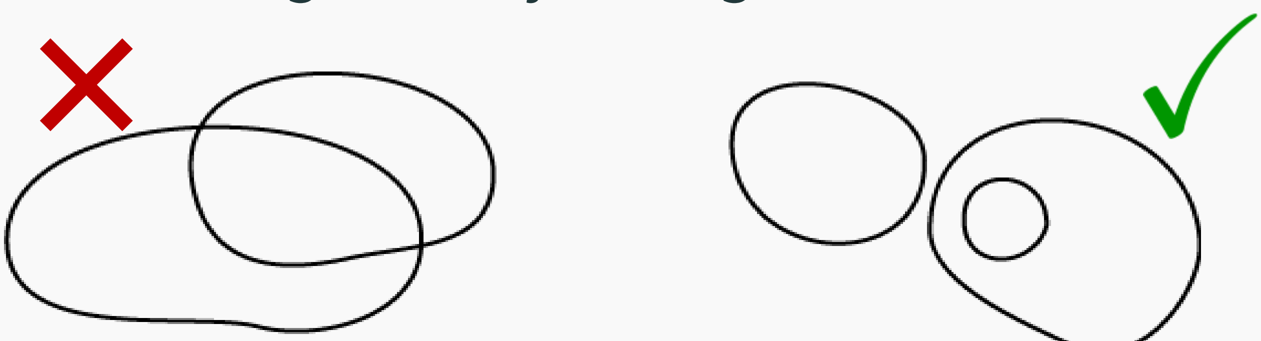
Current Limitations to Uniqueness Theorem

- Gap between necessary and sufficient # of samples:



Sufficient Necessary

- Restrictions on geometry of edge sets: *non-intersecting*



$\{\mu = 0\}$ $\{\mu = 0\}$

Uniqueness of signal (given edge set)

Theorem: If **f** is PWC* with edge set $\mathbf{E} = \{\mu = \mathbf{0}\}$ with μ minimal and bandlimited to $\mathbf{\Lambda}$ then $\mathbf{g} = \mathbf{f}$ is the unique solution to

$$\mu \cdot \nabla \mathbf{g} = \mathbf{0} \ \text{ s.t. } \ \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{g}}[\mathbf{k}], \mathbf{k} \in \mathbf{\Gamma}$$

when the sampling set $\mathbf{\Gamma} \supseteq 3\mathbf{\Lambda}$

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when the sampling set $\mathbf{\Gamma} \supseteq 3\mathbf{\Lambda}$

*Some geometric restrictions apply

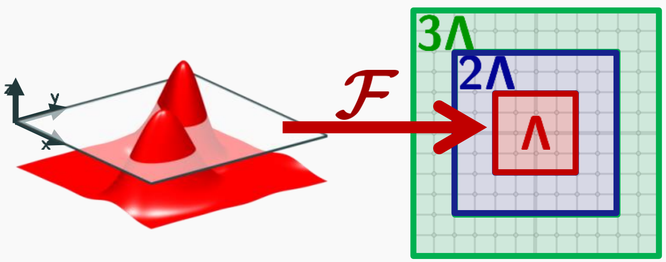
Equivalently,

$$\mathbf{f} = \arg\min_{\mathbf{g}} ||\mu \cdot \nabla \mathbf{g}|| \quad \text{s.t.} \quad \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{g}}[\mathbf{k}], \mathbf{k} \in \mathbf{\Gamma}$$

Summary of Proposed *Off-the-Grid Framework*

- Extend Prony/FRI methods to recover multidimensional singularities (curves, surfaces)

- Unique recovery from *uniform* Fourier samples: # of samples proportional to degree of edge set polynomial



- Two-stage recovery
  1. Recover edge set by solving linear system
  2. Recover amplitudes

Summary of Proposed *Off-the-Grid Framework*

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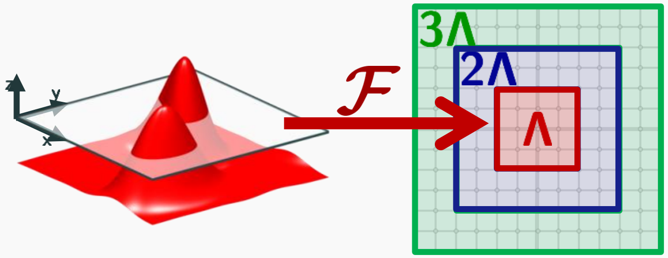
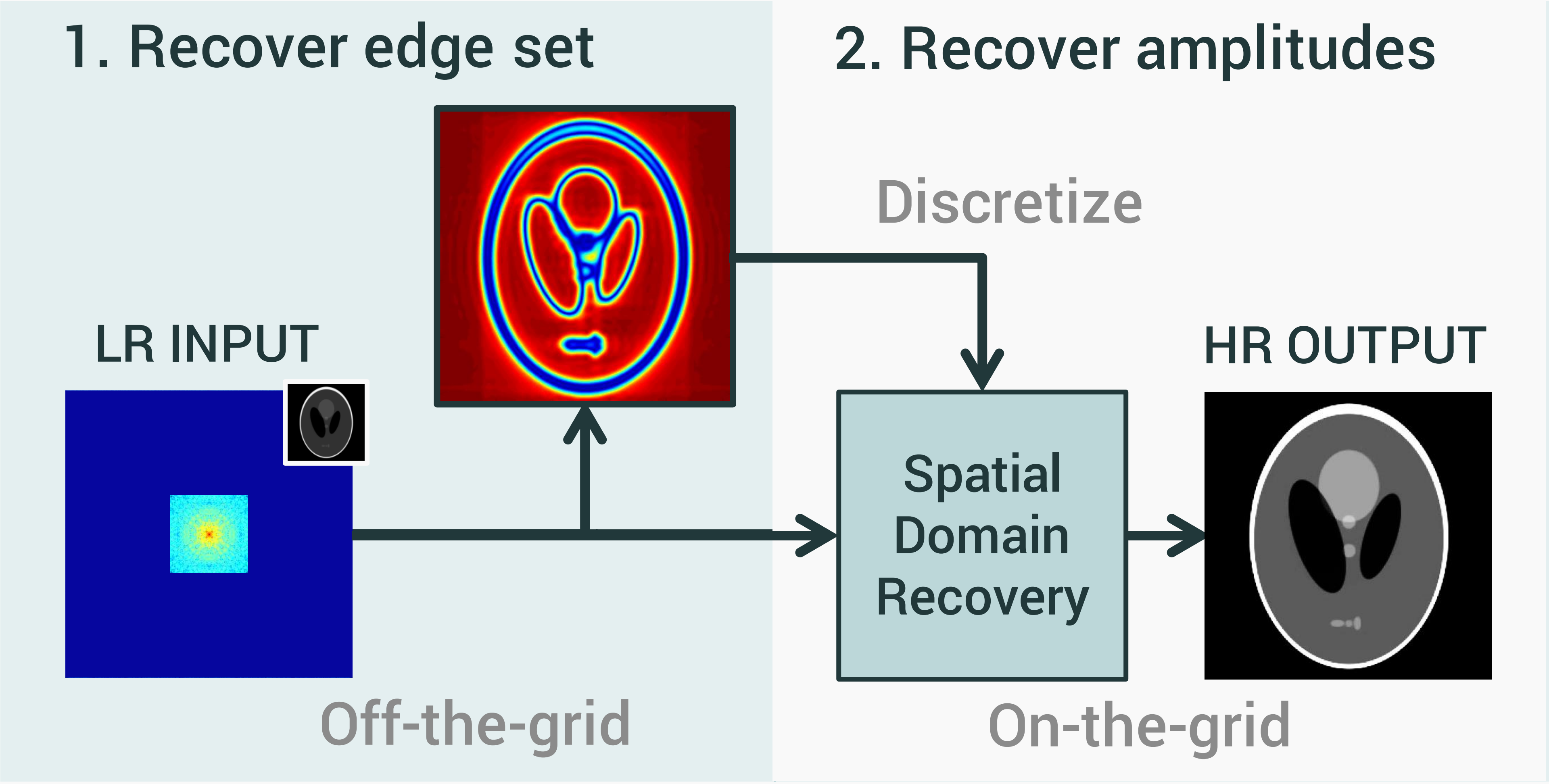


- Two-stage recovery
  1. Recover edge set by solving linear system (Robust?)
  2. Recover amplitudes (How?)

New *Off-the-Grid* Imaging Framework: *Algorithms*

Two-stage Super-resolution MRI Using Off-the-Grid Piecewise Constant Signal Model [O. & Jacob, ISBI 2015]

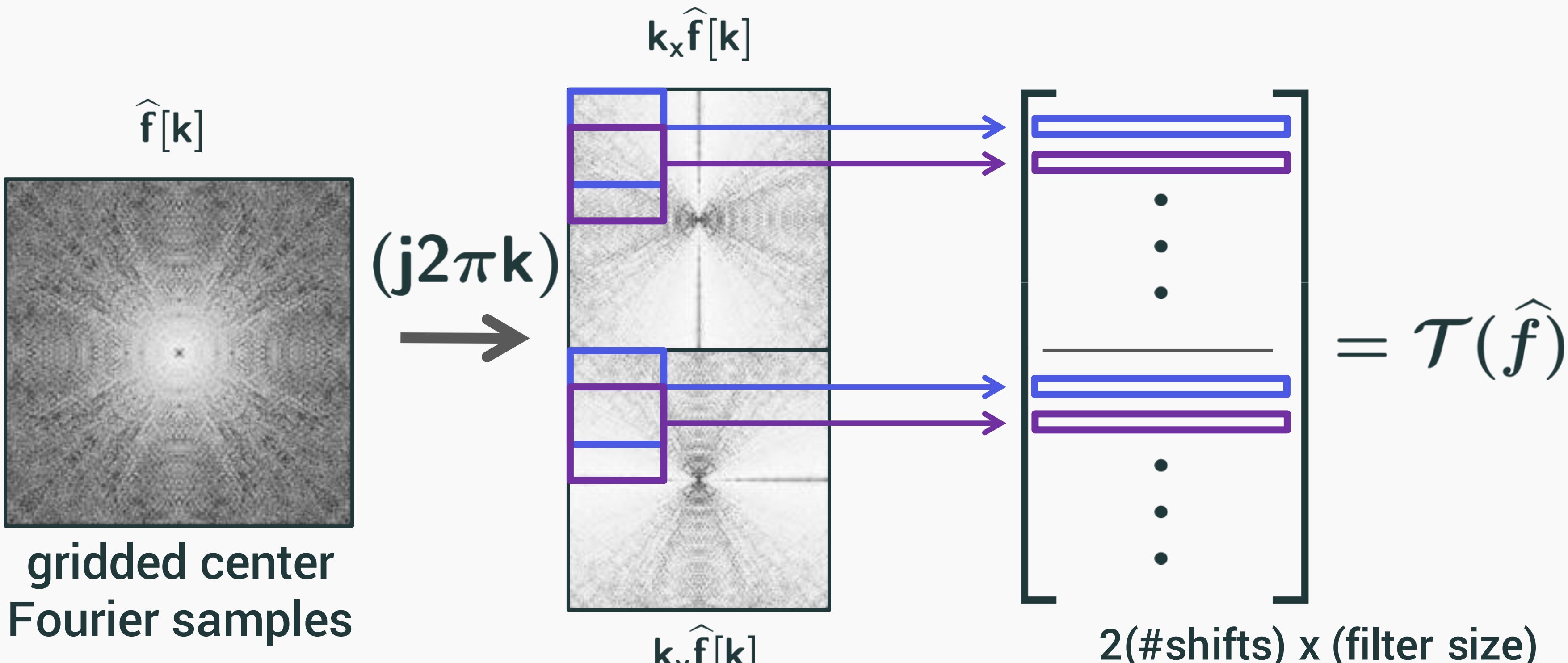


Matrix representation of annihilation

$$\mathcal{T}(\widehat{f})\,\boldsymbol{c} = 0$$

2-D convolution matrix
(block Toeplitz)

vector of filter coefficients

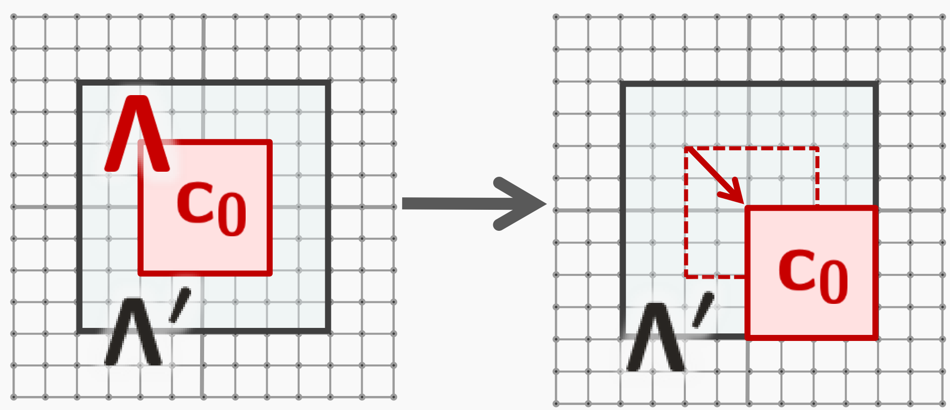


Basis of algorithms:
Annihilation matrix is low-rank

Prop: If the level-set function is bandlimited to Λ and the assumed filter support $\Lambda' \supset \Lambda$ then

$$\mathrm{rank}[\mathcal{T}(\widehat{\mathbf{f}})] \leq |\Lambda'| - (\#\text{shifts } \Lambda \text{ in } \Lambda')$$

Fourier domain



Spatial domain $\mu(x, y) \longrightarrow e^{j2\pi(kx+ly)}\mu(x, y)$

Basis of algorithms:
Annihilation matrix is low-rank

Prop: If the level-set function is bandlimited to Λ and the assumed filter support $\Lambda' \supset \Lambda$ then

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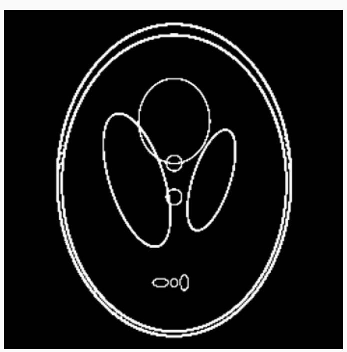
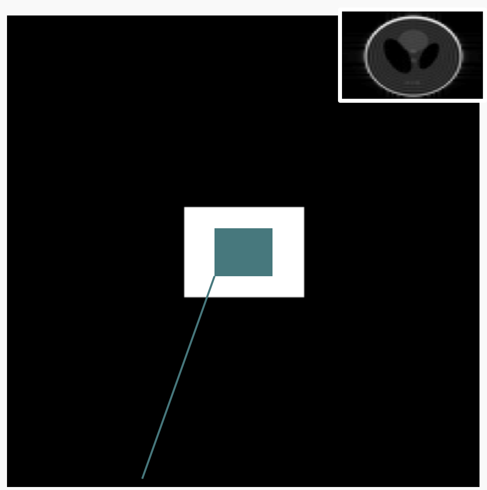
Example:
Shepp-Logan



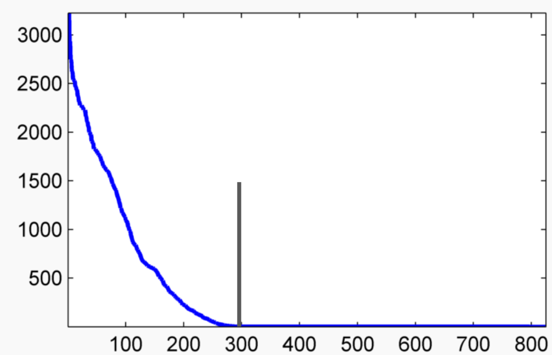
Fourier domain



Assumed filter: 33x25

Samples: 65x49

$$\sigma(\mathcal{T}(\widehat{f}))$$



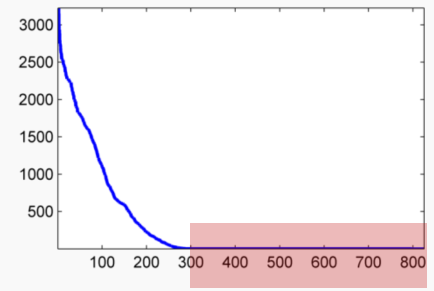
Rank $\approx$ 300

Stage 1: Robust annihilting filter estimation

1. Compute SVD

$$\mathcal{T}(\widehat{\mathbf{f}}) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

$$\sigma(\mathcal{T}(\widehat{\mathbf{f}}))$$

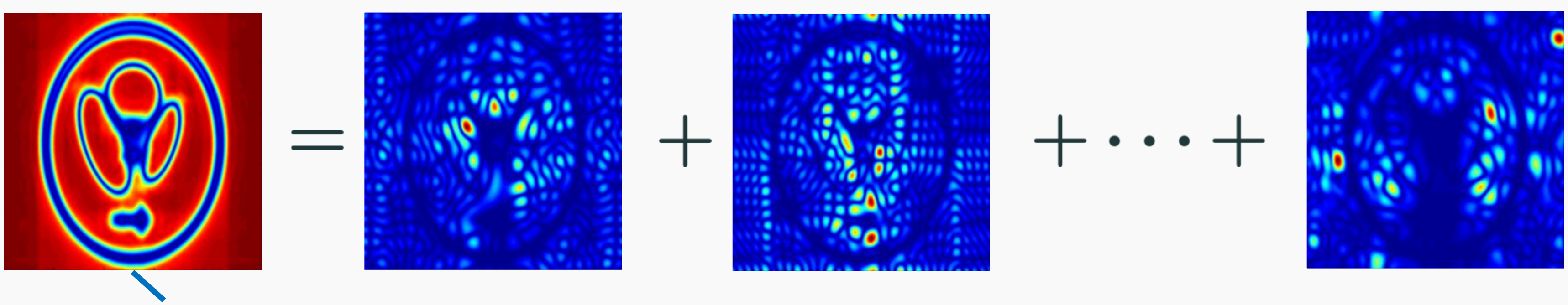


2. Identify **null space**

$$\mathbf{V} = [\mathbf{V}_S \ \mathbf{V}_N], \qquad \mathbf{V}_N = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$$

3. Compute sum-of-squares average

$$\mu = |\mathcal{F}^{-1}\mathbf{c}_1|^2 + |\mathcal{F}^{-1}\mathbf{c}_2|^2 + \cdots + |\mathcal{F}^{-1}\mathbf{c}_n|^2$$



Recover common zeros

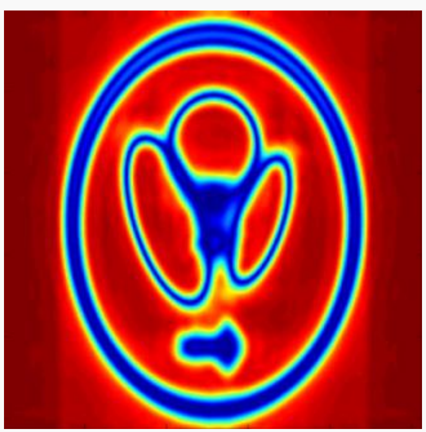
Stage 2: Weighted TV Recovery

$$\mathbf{f} = \arg\min_{\mathbf{g}} \|\mu \cdot \nabla \mathbf{g}\|_1 \quad \text{s.t.} \quad \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{g}}[\mathbf{k}], \mathbf{k} \in \Gamma$$

discretize

relax

$$\min_{x} \sum_{i} w_i \cdot |(Dx)_i| + \lambda \|Ax - b\|^2$$



Edge weights

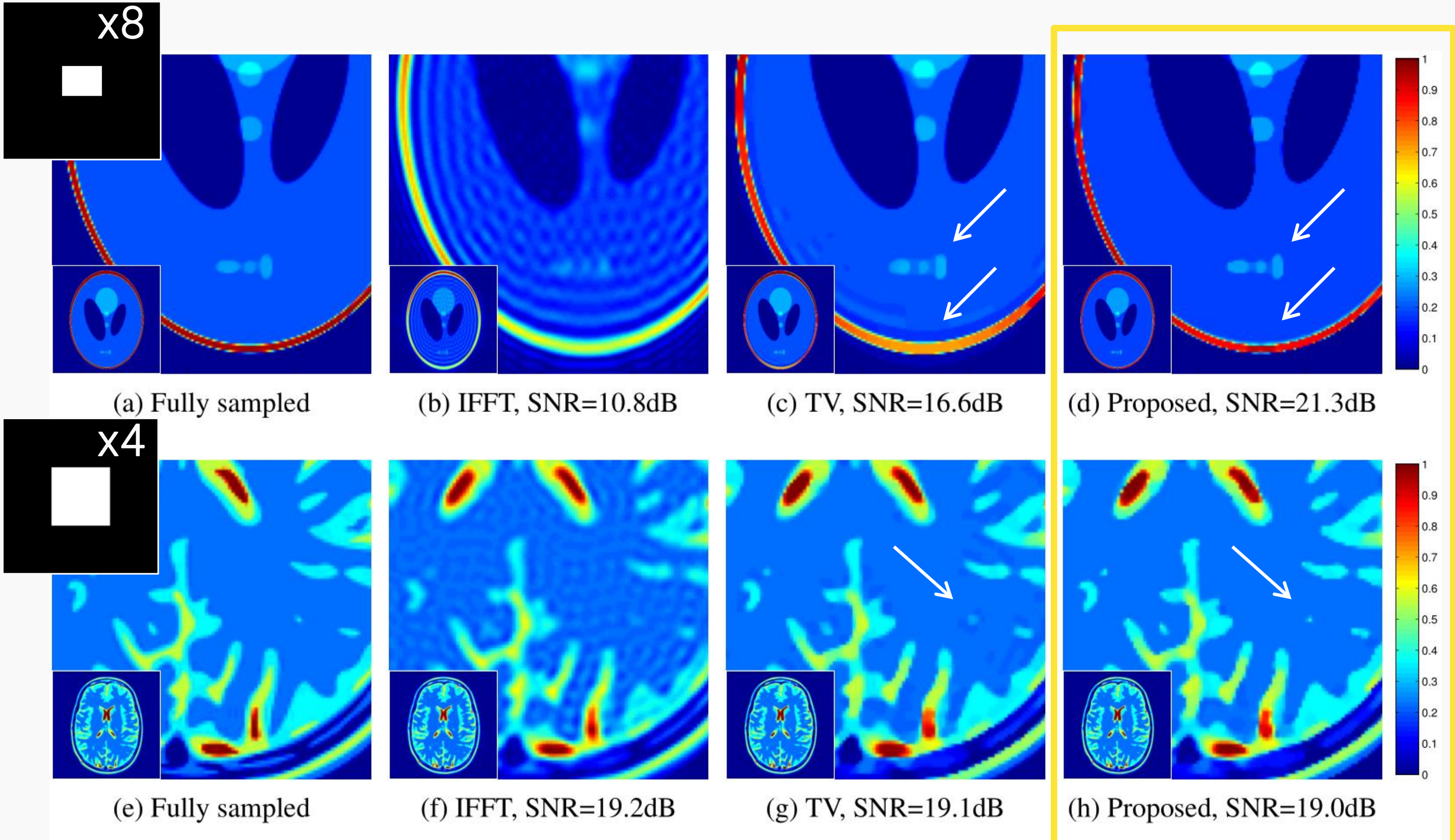
x = discrete spatial domain image

D = discrete gradient

A = Fourier undersampling operator

b = k-space samples

Super-resolution of MRI Medical Phantoms

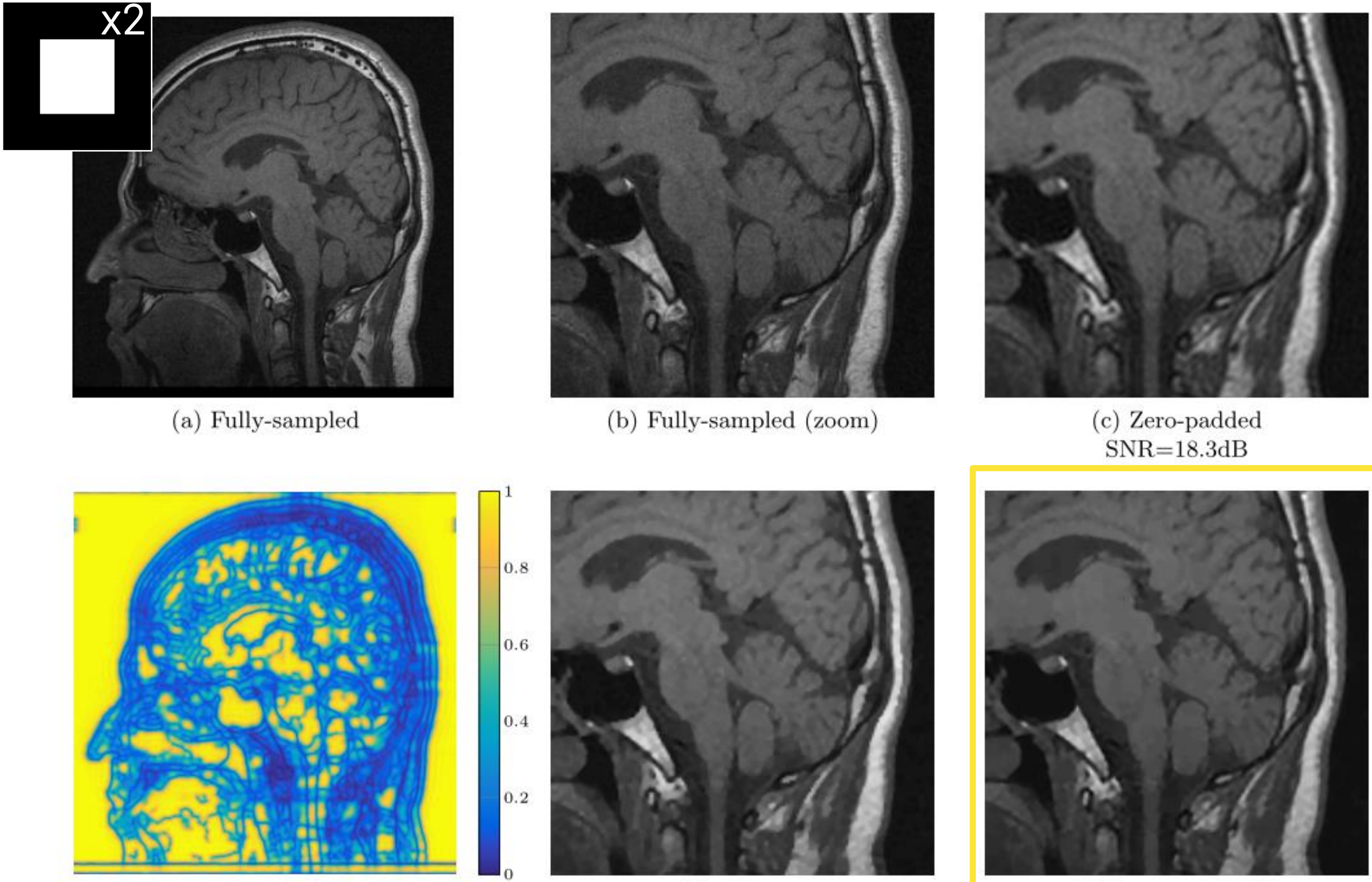


(a) Fully sampled (b) IFFT, SNR=10.8dB (c) TV, SNR=16.6dB (d) Proposed, SNR=21.3dB

(e) Fully sampled (f) IFFT, SNR=19.2dB (g) TV, SNR=19.1dB (h) Proposed, SNR=19.0dB

Analytical phantoms from *[Guerquin-Kern, 2012]*

Super-resolution of Real MRI Data



(a) Fully-sampled

(b) Fully-sampled (zoom)

(c) Zero-padded
SNR=18.3dB

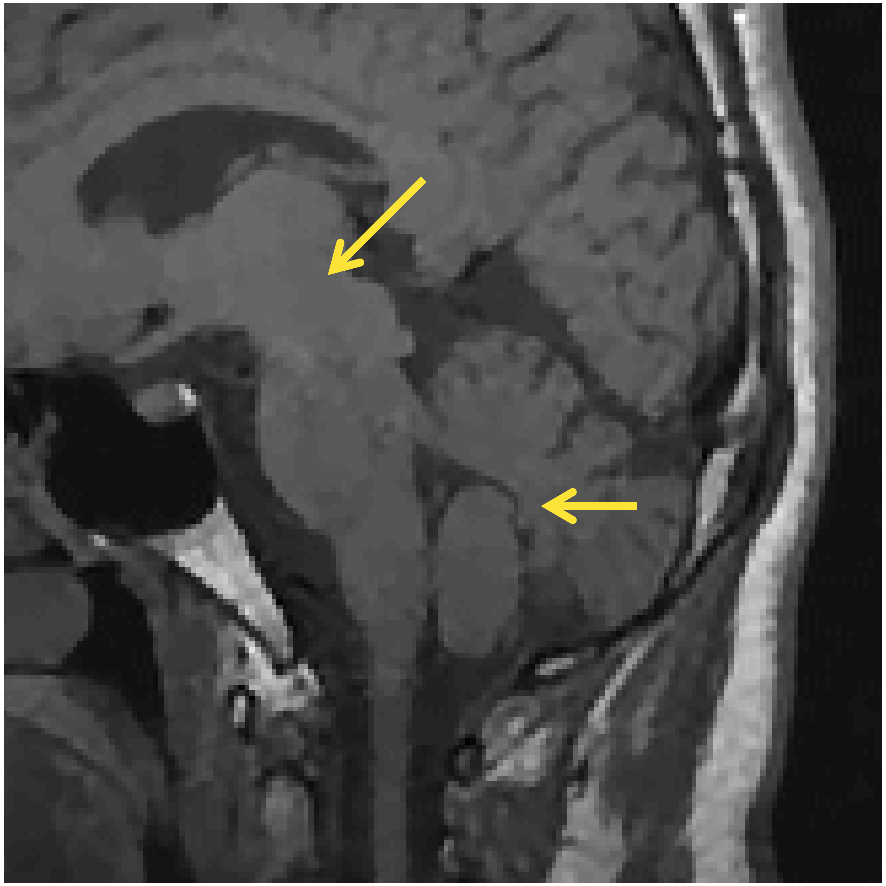
(d) Edge set estimate
(65×65 coefficients)

(e) TV reg.
SNR=18.5dB

(f) Proposed, LSLP
SNR=18.9dB

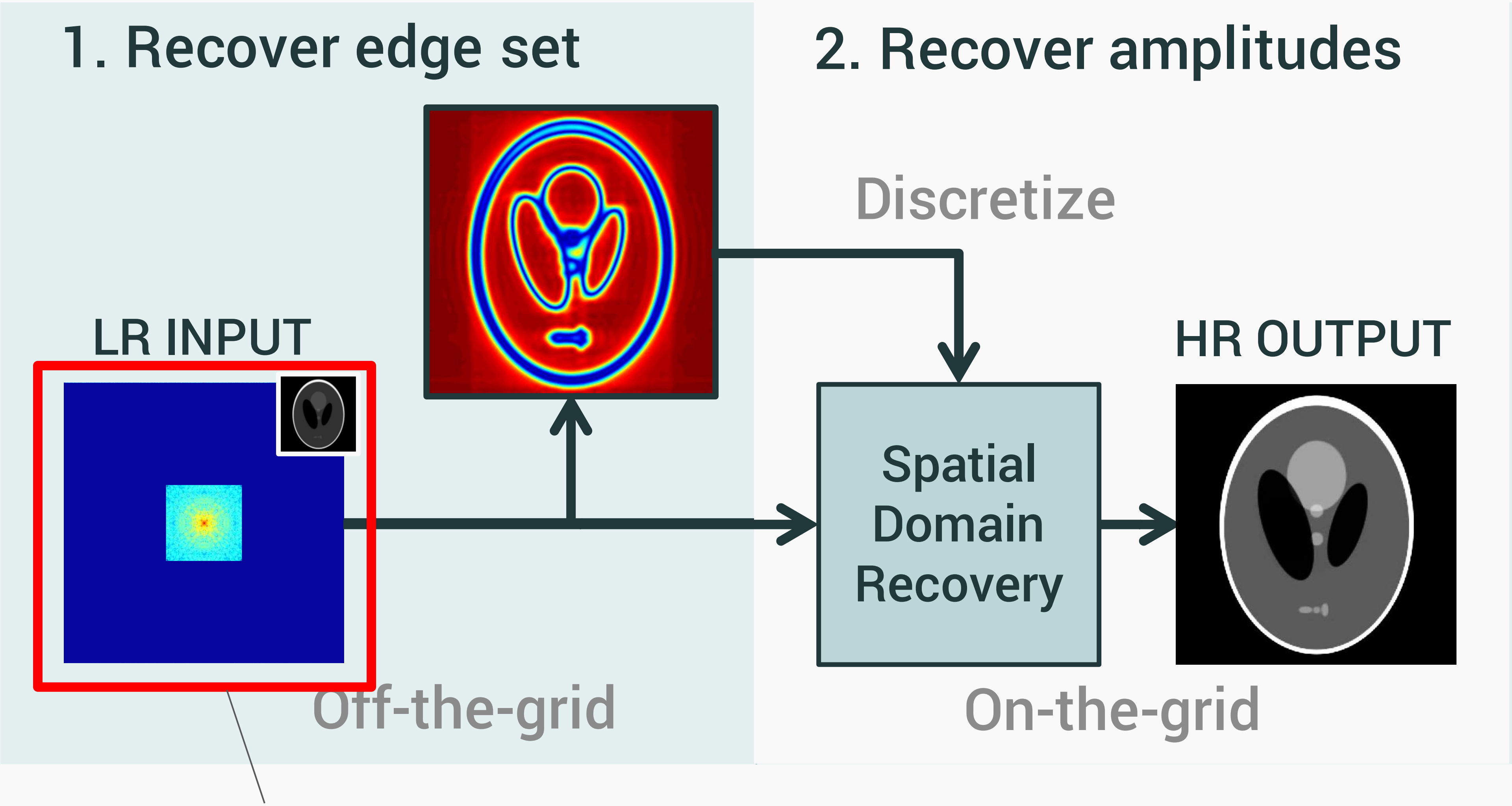
Super-resolution of Real MRI Data (Zoom)

(e) TV reg.
SNR=18.5dB



(f) Proposed, LSLP
SNR=18.9dB

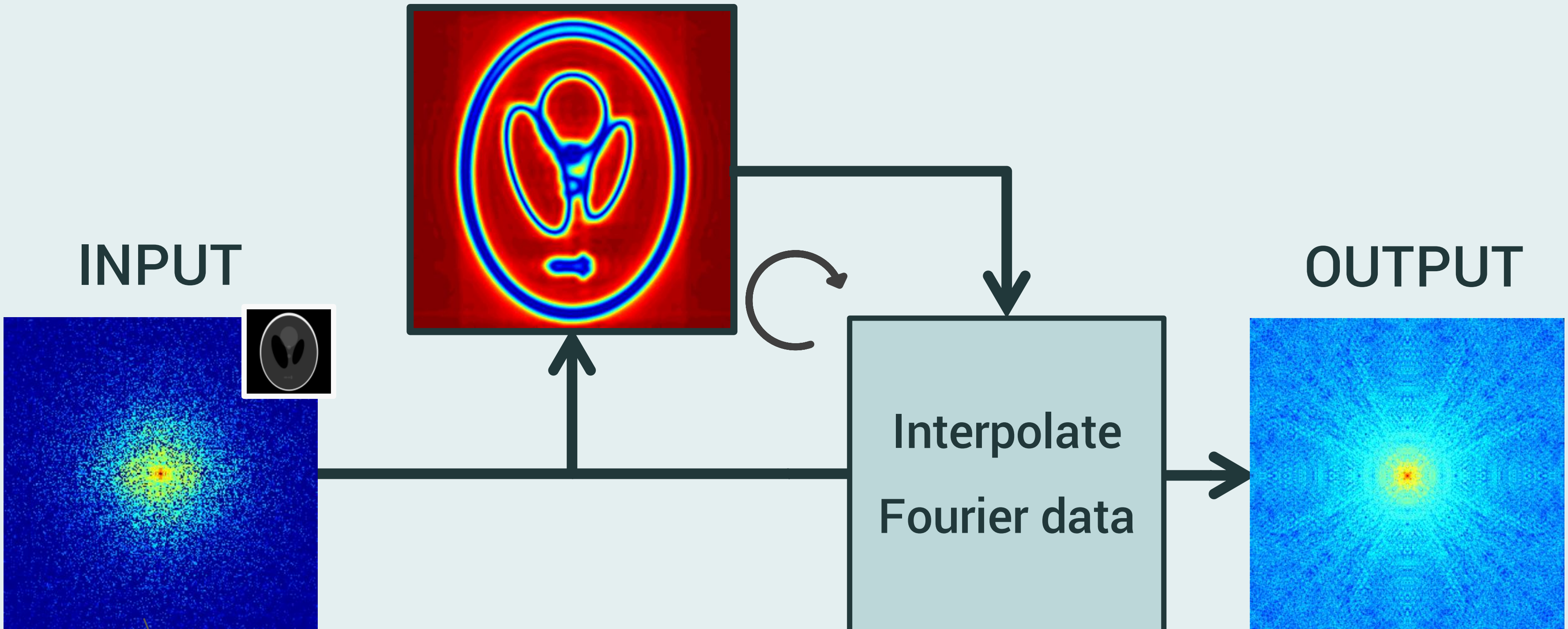
Two Stage Algorithm



Need uniformly sampled region!

One Stage Algorithm [O. & Jacob, SampTA 2015]

Jointly estimate edge set and amplitudes



Off-the-grid

Accommodate random samples

Pose recovery as a one-stage structured low-rank matrix completion problem

Recall: $\mathcal{T}(\widehat{\mathbf{f}})$ low rank $\leftrightarrow$ $\mathbf{f}$ piecewise constant

Toeplitz-like matrix built from Fourier data

Pose recovery as a one-stage structured low-rank matrix completion problem

$$\min_{\widehat{\mathbf{f}}} \quad \mathrm{rank}[\mathcal{T}(\widehat{\mathbf{f}})] \quad \textbf{s.t.} \quad \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{b}}[\mathbf{k}], \mathbf{k} \in \boldsymbol{\Gamma}$$

Pose recovery as a one-stage
structured low-rank matrix completion problem

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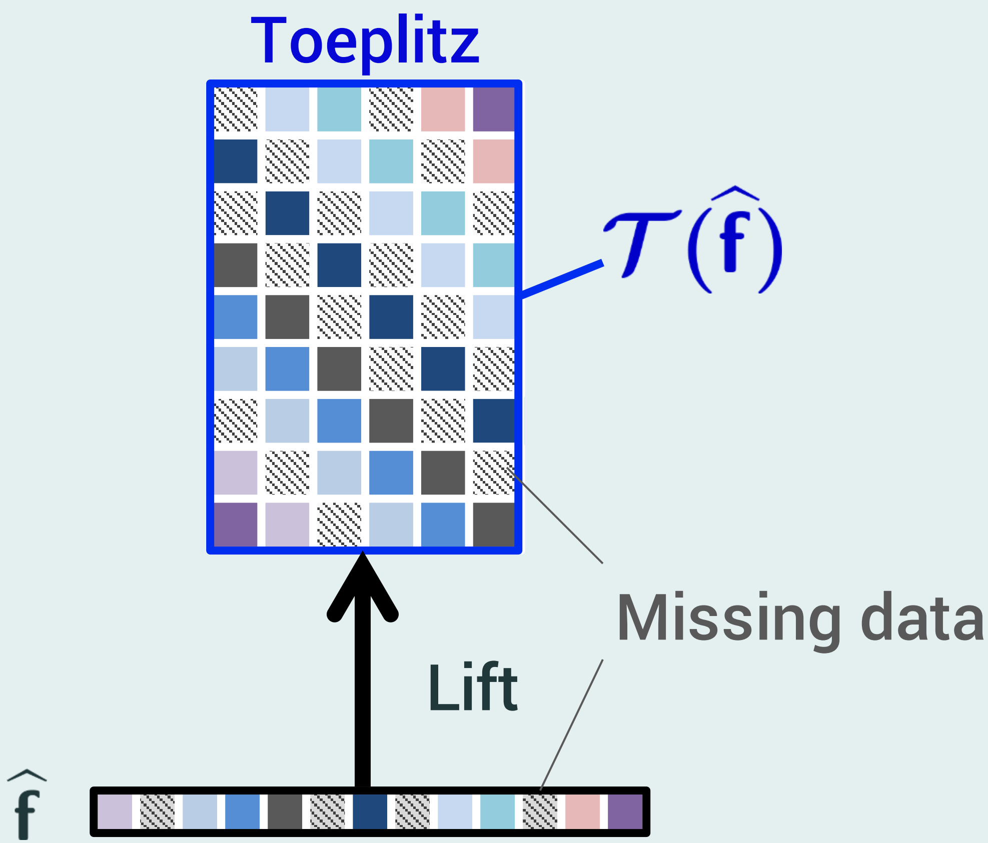
1-D Example:



Pose recovery as a one-stage structured low-rank matrix completion problem

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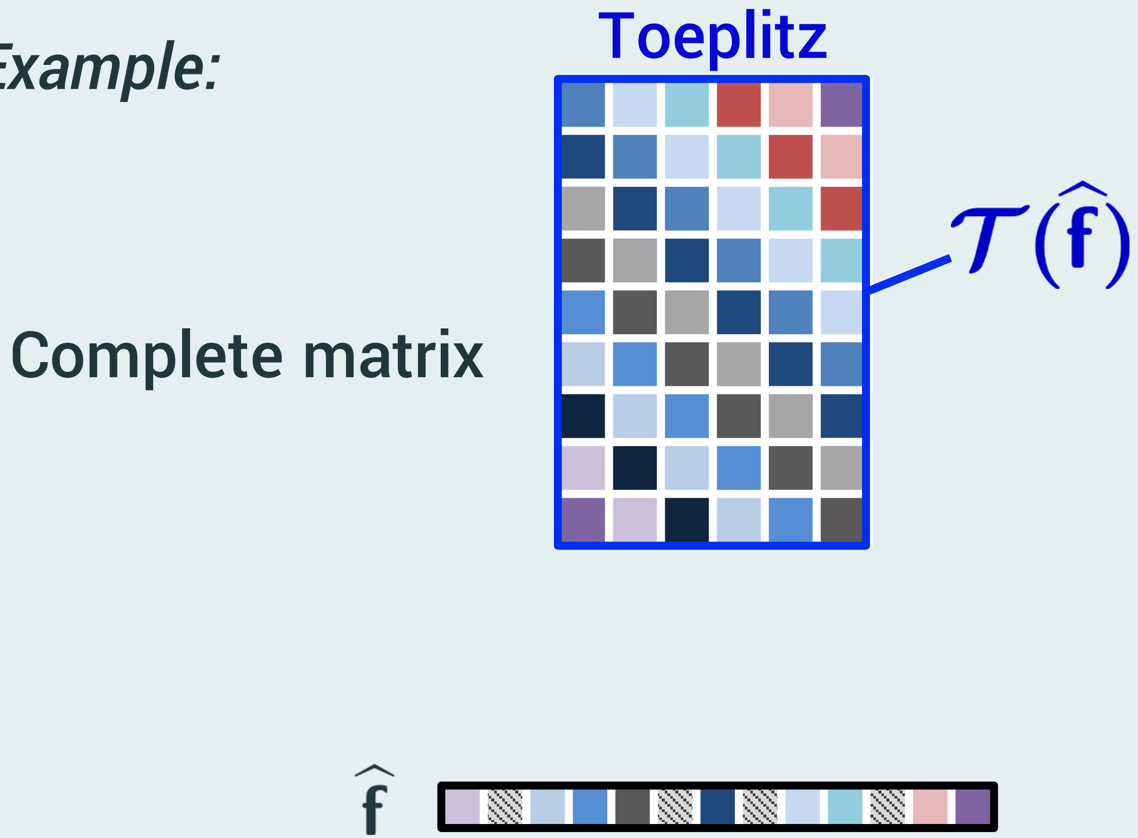
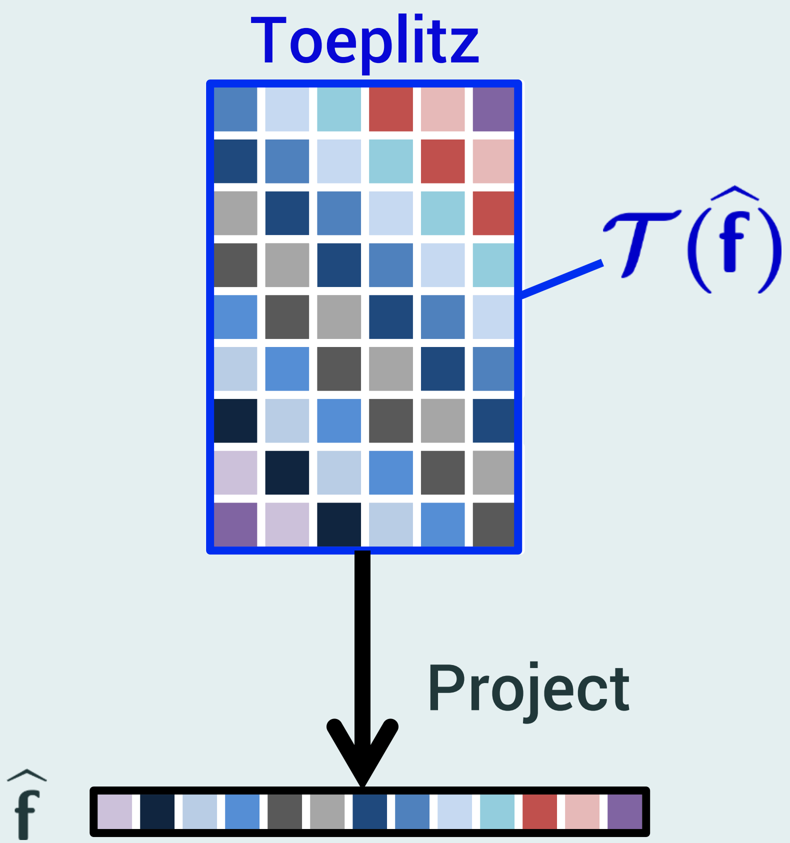
1-D Example:



Pose recovery as a one-stage **structured low-rank matrix completion** problem

$$\min_{\widehat{\mathbf{f}}} \quad \mathrm{rank}[\mathcal{T}(\widehat{\mathbf{f}})] \quad \textbf{s.t.} \quad \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{b}}[\mathbf{k}], \mathbf{k} \in \boldsymbol{\Gamma}$$

1-D Example:



Pose recovery as a one-stage
structured low-rank matrix completion problem

$$\min_{\widehat{\mathbf{f}}} \quad \mathrm{rank}[\mathcal{T}(\widehat{\mathbf{f}})] \quad \textbf{s.t.} \quad \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{b}}[\mathbf{k}], \mathbf{k} \in \boldsymbol{\Gamma}$$

NP-Hard!

Pose recovery as a one-stage
structured low-rank matrix completion problem

$$\min_{\widehat{\mathbf{f}}} \quad \mathrm{rank}[\mathcal{T}(\widehat{\mathbf{f}})] \ \ \textbf{s.t.} \ \ \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{b}}[\mathbf{k}], \mathbf{k} \in \mathbf{\Gamma}$$

$$\Big\downarrow \ \textit{Convex Relaxation}$$

$$\min_{\widehat{\mathbf{f}}} \quad \|\mathcal{T}(\widehat{\mathbf{f}})\|_* \ \ \textbf{s.t.} \ \ \widehat{\mathbf{f}}[\mathbf{k}] = \widehat{\mathbf{b}}[\mathbf{k}], \mathbf{k} \in \mathbf{\Gamma}$$

Nuclear norm – sum of singular values

Computational challenges

- **Standard algorithms are slow:**

  Apply ADMM = Singular value thresholding (SVT)

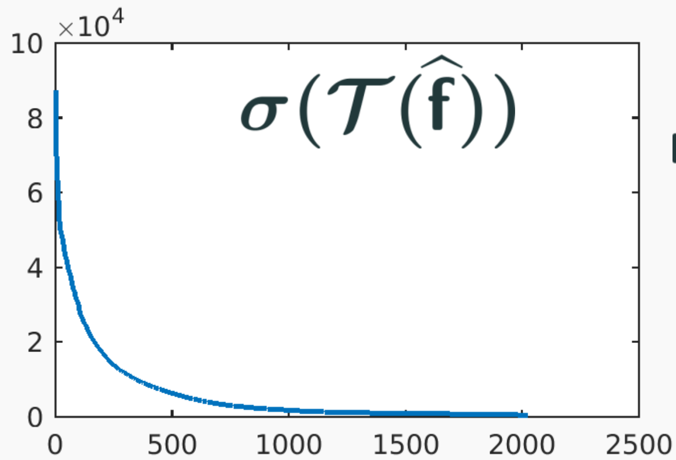
  Each iteration requires a large SVD:

  $$\mathbf{dim}(\mathcal{T}(\widehat{\mathbf{f}})) \approx \text{(\#pixels) x (filter size)}$$

  e.g. 10^6 x 2000

- **Real data can be "high-rank":**

e.g.
Singular values of
Real MR image



$$\mathbf{rank}(\mathcal{T}(\widehat{\mathbf{f}})) \approx 1000$$

Proposed Approach: Adapt IRLS algorithm

- IRLS: Iterative Reweighted Least Squares
- Proposed for low-rank matrix completion in [Fornasier, Rauhut, & Ward, 2011], [Mohan & Fazel, 2012]
- Adapt to structured matrix case:

$$\begin{cases} \mathbf{W} \leftarrow [\mathcal{T}(\widehat{\mathbf{f}})^*\mathcal{T}(\widehat{\mathbf{f}}) + \epsilon\mathbf{I}]^{-\frac{1}{2}} & \text{(weight matrix update)} \\ \widehat{\mathbf{f}} \leftarrow \arg\min_{\widehat{\mathbf{f}}} \|\mathcal{T}(\widehat{\mathbf{f}})\mathbf{W}^{\frac{1}{2}}\|_F^2 \ \text{ s.t. } \ \mathbf{P}\widehat{\mathbf{f}} = \mathbf{b} & \text{(LS problem)} \end{cases}$$

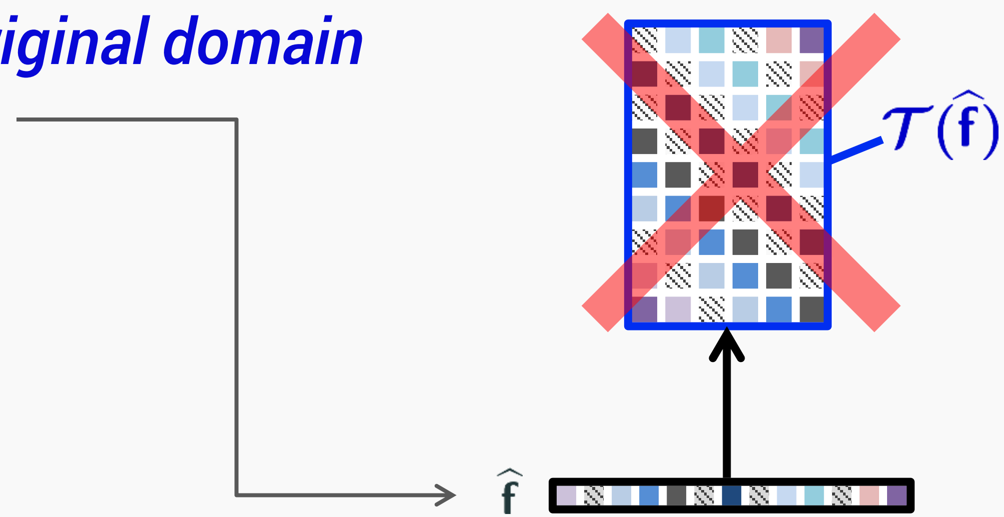
- Without modification, this approach is slow!

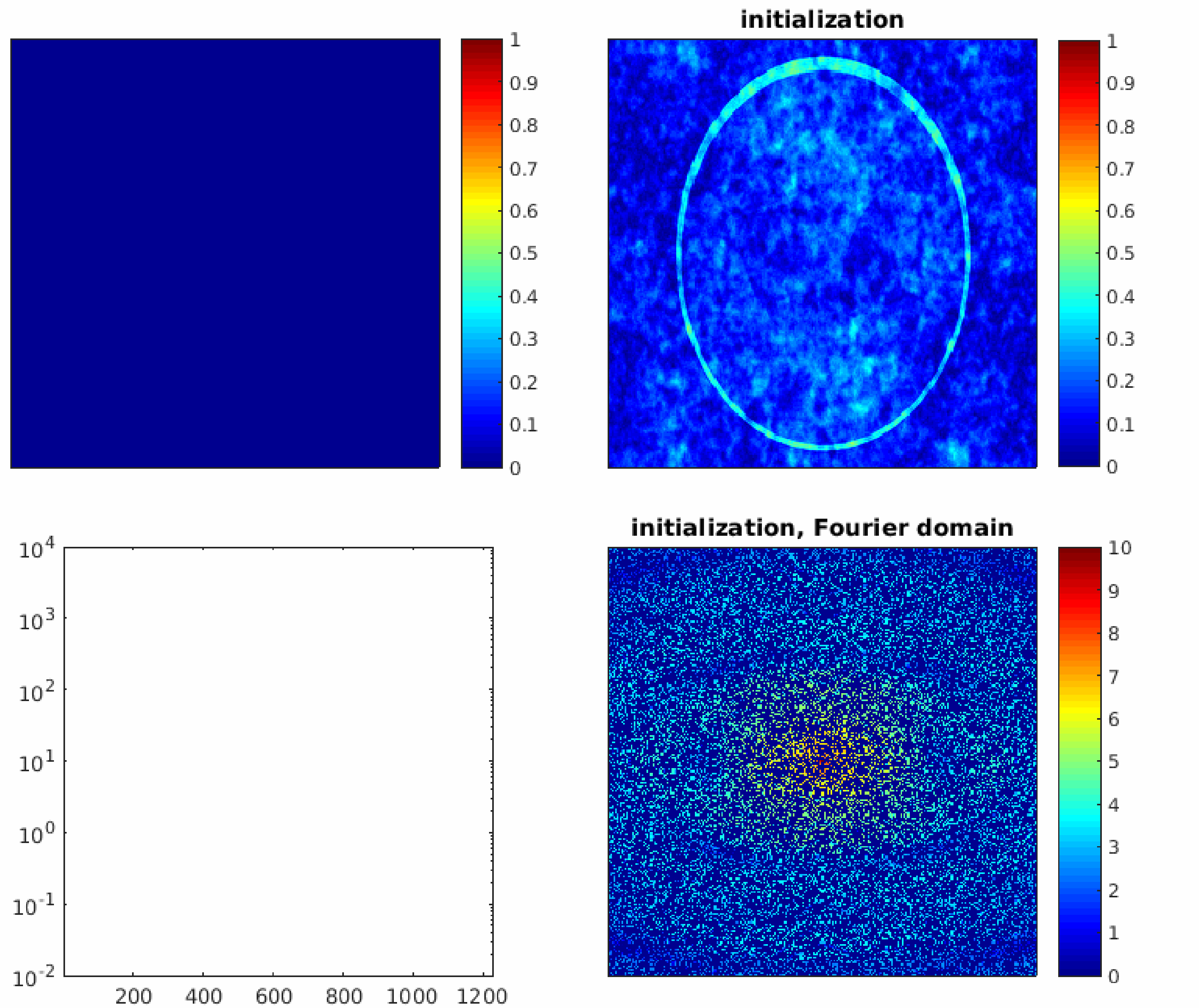
GIRAF algorithm [O. & Jacob, ISBI 2016]

- **GIRAF = Generic Iterative Reweighted Annihilating Filter**
- Exploit convolution structure to simplify IRLS algorithm:

$$\begin{cases} \mu \leftarrow \sum \lambda_i^{-\frac{1}{2}} \mu_i \quad \text{(annihilating filter update)} \\ \widehat{\mathbf{f}} \leftarrow \arg\min_{\widehat{\mathbf{f}}} \|\widehat{\mathbf{f}} * \widehat{\mu}\|_2^2 \quad \text{s.t.} \quad \mathbf{P}\widehat{\mathbf{f}} = \mathbf{b} \quad \text{(LS problem)} \end{cases}$$

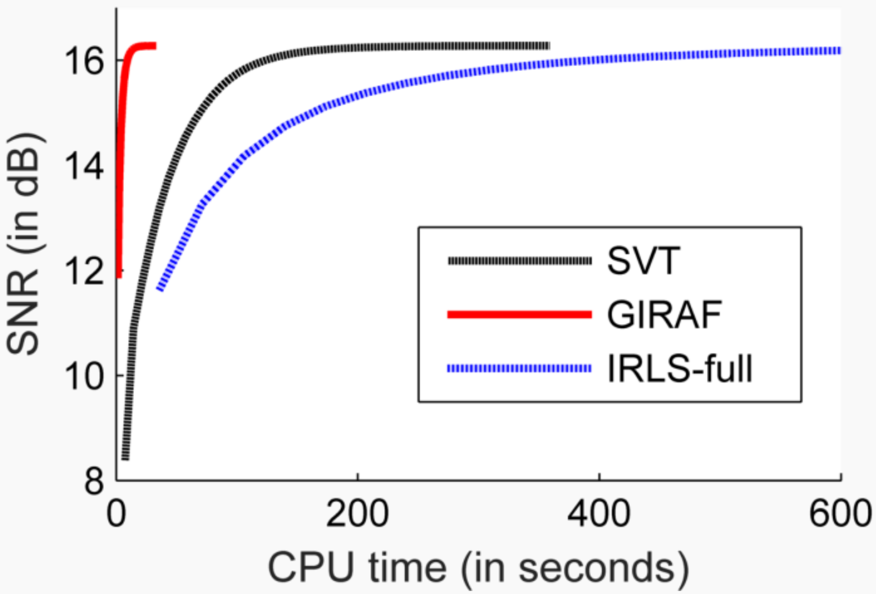
- Condenses weight matrix to *single* annihilating filter
- Solves problem in *original domain*





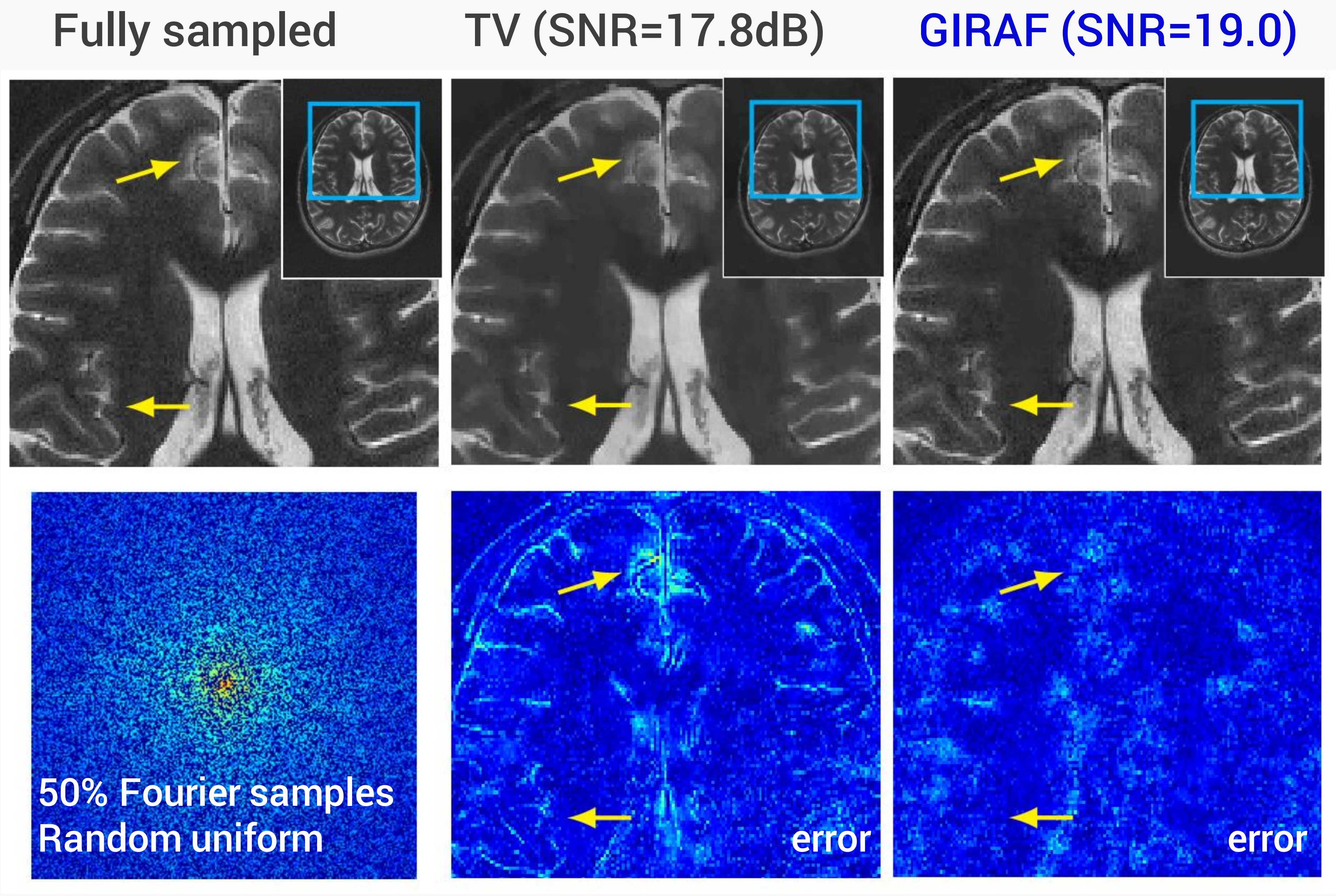
initialization
initialization, Fourier domain

Convergence speed of GIRAF



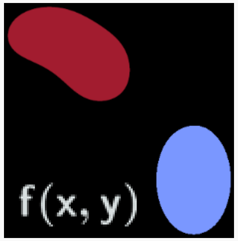
	15×15 filter		31×31 filter	
Algorithm	# iter	total	# iter	total
SVT	7	110s	11	790 s
GIRAF	6	20s	7	44 s

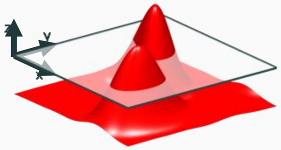
Table: iterations/CPU time to reach convergence tolerance of NMSE < 10^{-4}.


Fully sampled
TV (SNR=17.8dB)
GIRAF (SNR=19.0)
50% Fourier samples
Random uniform
error
error

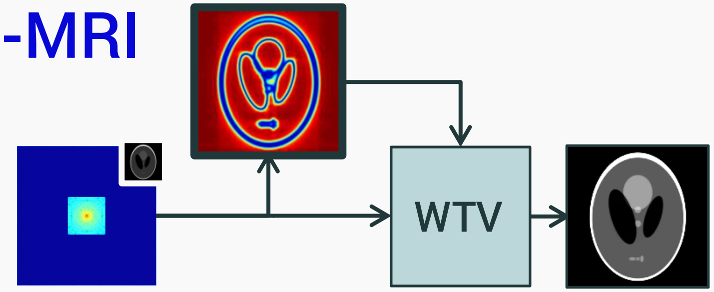
Summary

- New framework for off-the-grid image recovery
  - Extends FRI annihilating filter framework to piecewise polynomial images
  - Sampling guarantees





- Two stage recovery scheme for SR-MRI
  - Robust edge mask estimation
  - Fast weighted TV algorithm

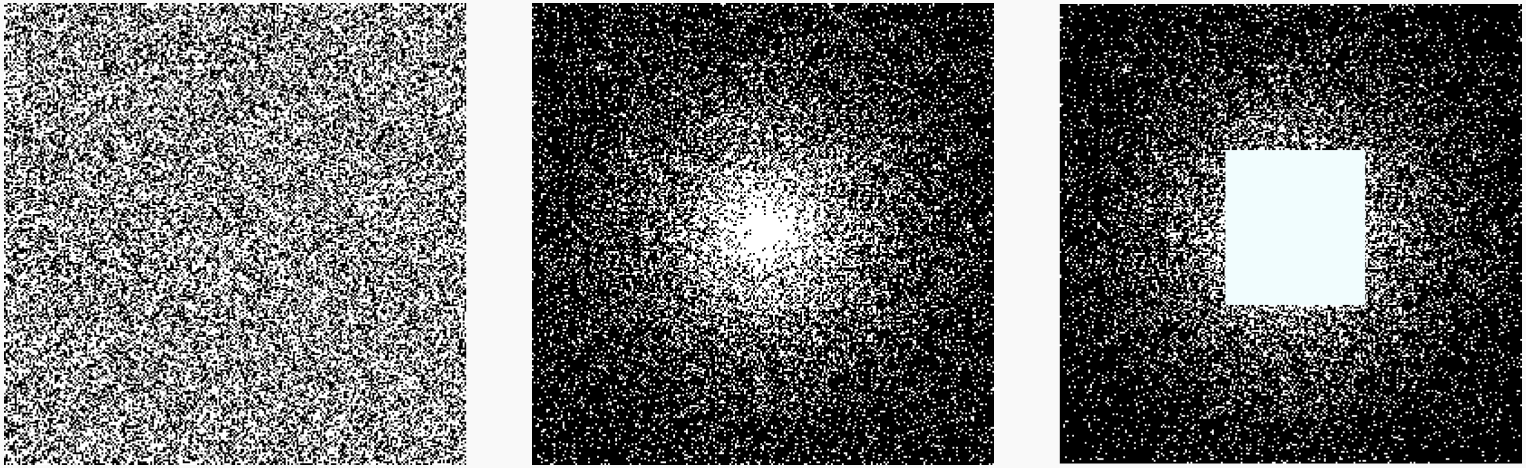


- One stage recovery scheme for CS-MRI
  - Structured low-rank matrix completion
  - Fast GIRAF algorithm

$$\min_{\hat{\mathbf{f}}} \|\mathcal{T}(\hat{\mathbf{f}})\|_*$$

Future Directions

- **Focus:** One stage recovery scheme for CS-MRI
  - Structured low-rank matrix completion $$\min_{\widehat{\mathbf{f}}} \|\mathcal{T}(\widehat{\mathbf{f}})\|_*$$
- Recovery guarantees for random sampling?
- What is the optimal random sampling scheme?



Thank You!

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